On the integrability of twofold 1 : 2 Hamiltonian resonance

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Outline:

In a recent paper Mazrooei-Sebdani et al, 2021 study a particular case of 1 : 2 : 1 : 2 Hamiltonian normal form, truncated to order 3. They find several cases of integrability. The main purpose of this work is to find other integrable cases.

1. Introduction, motivation and the main result

2. Simplification of the normal form

3. Concluding remarks

Some Relevant References

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1. Introduction, motivation and the main result

Consider a real analytic Hamiltonian $H(x,y), (x,y) \in \mathbb{R}^{2n}$. In a neighborhood of equilibrium $(x,y) = (0,0)$ it can be represented in the form

$$H = H_2 + H_3 + \ldots + H_j + \ldots, \qquad (1)$$

where $H_2 = \sum \omega_j (x_j^2 + y_j^2)$, $\omega_j > 0$ and H_j are homogeneous polynomials of degree j.

It is called that the frequency vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ satisfies a resonance of order $\mathbf{k}$ if there exist integers k_j such that $\sum k_j \omega_j = 0$ and $\sum |k_j| = \mathbf{k}$.

Recall the standard symplectic form in $\mathbb{R}^{2n}$ and associated Poisson bracket

$$\omega^2 = \sum_j dy_j \wedge dx_j, \quad \{F, G\} = \sum_j \frac{\partial F}{\partial y_j}\frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial y_j},$$

where F, G are smooth functions.

Using canonical near-identity transformation Arnold et al, 2006 (i.e., transformation which preserves the symplectic form), the Hamiltonian function (1) can be simplified. This new simplified Hamiltonian $\overline{H}$ is called Birkhoff normal form, and in the presence of resonances – resonant Birkhoff normal form or Birkhoff-Gustavson normal form. In different words, if we introduce the so called action-angle variables

$$x_j = \sqrt{2I_j}\cos\phi_j, \quad y_j = \sqrt{2I_j}\sin\phi_j, \quad j = 1, \ldots, n \tag{2}$$

the Hamiltonian normal form in the absence of angle variables is called Birkhoff normal form.

Often, for better understanding the behavior of the studied Hamiltonian system in a neighborhood of the equilibrium, one considers the normal form restricted to certain order

$$\overline{H} = H_2 + \overline{H}_3 + \ldots + \overline{H}_s. \tag{3}$$

It is known that the truncated to any order Birkhoff normal form is integrable Arnold et al, 2006. Notice that by construction H_2 and $\overline{H}_j$ Poisson commute, that is, $\{\overline{H}_j, H_2\} = 0$. Hence, the truncated resonant Birkhoff normal form always admits at least two first integrals H_2 and $\overline{H}$.

It is important to have knowledge for the first integrals of the truncated resonant normal form (if they exist), because they are approximate integrals for the original system (we refer to Verhulst, 1998 for a detailed discussion). That is why when the truncated resonant normal form turns out to be integrable, the initial Hamiltonian system is called near-integrable.

Recall that a real Hamiltonian system, defined on real symplectic manifold of dimension $2n$ is said to be integrable in sense of Liouville (or completely integrable) if there exist n independent (almost everywhere) first integrals in involution. Due to Liouville-Arnold theorem (Arnold et al, 2006) almost whole phase space is foliated by closed invariant manifold, called Lagrangian manifolds with dimension n. In other words, the trajectories of completely integrable system behave in a regular way- they stay on manifolds with dimension half of the dimension of the phase space.

On the other hand, the trajectories of a non-integrable Hamiltonian system, whatever is the reason for non-integrability share the same behavior: they no longer live on n-dimensional manifolds, but fill up entire phase space. Apart from exact solvability, the integrable systems are important, because they facilitate the study of the dynamics of nearby Hamiltonian systems.

In general, the resonant Hamiltonian normal forms (even non-integrable) can be used to compute: normal modes and evaluate their stability, bifurcations of periodic orbits, invariant manifolds, associated to unstable normal modes, etc.

Let us review some known result on integrability of the resonant normal forms. Since the two degrees of freedom truncated normal forms are integrable, the study of the integrability has shifted to three degrees of freedom resonant normal forms truncated to order 3. For complete integrability one needs one additional independent integral.

Firstly, van der Aa, 1983 proved that there is no additional quadratic or cubic first integral for $1:2:\omega$ resonances $\omega = 1, 3, 4$. Next, van der Aa and Verhulst, 1983 established a remarkable result that the Hamiltonian 1 : 2 : . . . : 2 resonance normal form truncated to order 3 is integrable for any number of degrees of freedom.

With a very original approach Duistermaat, 1984 proved that the 1 : 1 : 2 Hamiltonian resonant normal form does not admit additional analytic first integral. Along the way he discovered two non-trivial cases of integrability.

In an algebraic way described below, the genuine first order Hamiltonian resonances (that is, $1:2:\omega, \omega = 1, 3, 4$) are proven to be non-integrable except for the discrete values of the parameters, which are previously known in Christov, 2012.

For now less attention is paid on the study of integrability in Hamiltonian resonances with four or more degrees of freedom. Beyond the result in 1 : 2 : ... : 2 resonance mentioned above, a specific system that describes perturbed isotropic oscillators in 1 : 1 : 1 : 1 resonance is studied by Egea et al, 2011. After normalization and truncation this system turns out to integrable.

In a recent paper Mazrooei-Sebdani et al, 2021 study a particular case of 1 : 2 : 1 : 2 Hamiltonian normal form, truncated to order 3. They have found some integrable cases. In the same article they also examine the so called FWM (four-wave mixing) model, which has many applications in optics. This model is described by an 1 : 2 : 1 : 2 resonant integrable Hamiltonian (as a matter of fact, of fourth degree).

Our principal object here is to find more integrable cases in their normal form, a little bit rearranged to get Hamiltonian 1 : 1 : 2 : 2 resonance. The reason for that will be clear in a while.

Keeping $a_j \in \mathbb{R}$ and using $I_j = (x_j^2 + y_j^2)$, $j = 1, \ldots, 4$ introduced earlier in (2), we start with (compare with Mazrooei-Sebdani et al, 2021)

$$\overline{H} = H_2 + \overline{H}_3, \tag{4}$$

$$H_2 = \frac{1}{2}I_1 + \frac{1}{2}I_2 + I_3 + I_4 = \frac{1}{2}(x_1^2 + y_1^2) + \frac{1}{2}(x_2^2 + y_2^2) + (x_3^2 + y_3^2) + (x_4^2 + y_4^2), \quad (5)$$

$$\begin{aligned}\overline{H}_3 = {} & a_1[x_3(x_1^2 - y_1^2) + 2x_1y_1y_3] + a_3[x_4(x_1^2 - y_1^2) + 2x_1y_1y_4] \\ & + a_5[x_3(x_2^2 - y_2^2) + 2x_2y_2y_3] + a_7[x_4(x_2^2 - y_2^2) + 2x_2y_2y_4] \\ & + a_9[x_3(x_1x_2 - y_1y_2) + y_3(y_1x_2 + x_1y_2)] \\ & + a_{11}[x_4(x_1x_2 - y_1y_2) + y_4(y_1x_2 + x_1y_2)].\end{aligned} \quad (6)$$

As we pointed out above, the normal form always has two integrals $\overline{H}$ and H_2. Hence, for complete integrability we need to find two more independent integrals. Notice that if an additional independent integral exists, it must be a combination of the generators of the algebra of functions that commute with H_2.

First, it is worth recalling the list of integrable cases found in Mazrooei-Sebdani et al, 2021

1) $a_3 = a_5 = a_7 = a_9 = a_{11} = 0$ with integrals I_2 and I_4;

2) $a_1 = a_5 = a_7 = a_9 = a_{11} = 0$ with integrals I_2 and I_3;

3) $a_1 = a_3 = a_7 = a_9 = a_{11} = 0$ with integrals I_1 and I_4;

4) $a_1 = a_3 = a_5 = a_9 = a_{11} = 0$ with integrals I_1 and I_3;

5) $a_1 = a_3 = a_5 = a_7 = a_{11} = 0$ with integrals I_4 and $I_1 - I_2$;

6) $a_1 = a_3 = a_5 = a_7 = a_9 = 0$ with integrals I_3 and $I_1 + I_4$;

We call these case of integrability trivial.

The above list can be enlarged paying attention to the fact that the normal form (4) contains as "subsystems" two 1 : 2 : 2 cases which are integrable van der Aa and Verhulst, 1983, namely

7) $a_5 = a_7 = a_9 = a_{11} = 0$ with integrals I_2 and

$$K_1 = a_3^2(x_3^2 + y_3^2) + a_1^2(x_4^2 + y_4^2) - 2a_1a_3(x_3x_4 + y_3y_4). \tag{7}$$

Similarly,

8) $a_1 = a_3 = a_9 = a_{11} = 0$ with integrals I_1 and

$$K_2 = a_5^2(x_3^2 + y_3^2) + a_7^2(x_4^2 + y_4^2) - 2a_5a_7(x_3x_4 + y_3y_4). \tag{8}$$

Furthermore, it contains two 1 : 1 : 2 "subsystems" and consequently, we have two more integrable cases analogous to Duistermaat, 1984 ones

9) $a_3 = a_7 = a_9 = a_{11} = 0$ with integrals I_4 and

- $a_1 = a_5$ with additional integral

$$F_1 = x_1 y_2 - x_2 y_1. \tag{9}$$

- $a_1 = 2a_5$ with additional integral

$$G_1 = (x_1 y_2 - x_2 y_1)^2 (x_2^2 + y_2^2) + \frac{1}{2}[x_3(x_2^2 - y_2^2) + 2x_2 y_2 y_3]^2. \tag{10}$$

Similarly,

10) $a_1 = a_5 = a_9 = a_{11} = 0$ with integrals I_3 and

- $a_3 = a_7$ with additional integral F_1;

- $a_3 = 2a_7$ with additional integral

$$G_2 = (x_1y_2 - x_2y_1)^2(x_2^2 + y_2^2) + \frac{1}{2}[x_4(x_2^2 - y_2^2) + 2x_2y_2y_4]^2. \qquad (11)$$

Remark. A natural question is whether these integrals in the cases 7)-10) can be extended for $a_9 \neq 0$ or $a_{11} \neq 0$. Numerical experiments show that if $a_9 \neq 0$ or $a_{11} \neq 0$, these quantities are no longer first integrals. But this fact does not exclude the possibility of the existence of other conserved quantities.

The rest of the paper is devoted to the search of other integrable cases. We simplify further the normal form (4) to get

$$\begin{aligned} \overline{H} = {} & b_1[x_3(x_1^2 - y_1^2) + 2x_1y_1y_3] + b_2[x_3(x_2^2 - y_2^2) + 2x_2y_2y_3] \\ & + b_3[x_4(x_1^2 - y_1^2) + 2x_1y_1y_4] + b_4[x_4(x_2^2 - y_2^2) + 2x_2y_2y_4]. \end{aligned} \qquad (12)$$

This simplification will be given in some detail in the next section. For this normal form we immediately deduce several integrable cases.

The case $b_2 = b_4 = 0$ and the case $b_1 = b_3 = 0$ are clearly analogous to the the items 7) and 8) above.

Similarly, in the case $b_3 = b_4 = 0$ and for the case $b_1 = b_2 = 0$ we get the integrable cases which correspond to the items 9) and 10).

The main result here is the following

Theorem 1. Suppose the parameters b_j are not as above. Then the normal form (12) is integrable only if:

(i) $b_1 = b_2$ and $b_3 = b_4$ with additional integrals F_1 and

$$Q_1 = b_3^2(x_3^2 + y_3^2) + b_1^2(x_4^2 + y_4^2) - 2b_1b_3(x_3x_4 + y_3y_4); \qquad (13)$$

(ii) $b_1 = 2b_2$ and $b_3 = 2b_4$ with additional integrals

$$Q_2 = b_4^2(x_3^2 + y_3^2) + b_2^2(x_4^2 + y_4^2) - 2b_2b_4(x_3x_4 + y_3y_4). \tag{14}$$

and

$$\begin{aligned} G_3 = {} & b_2^2 G_1 + b_4^2 G_2 \\ & + \frac{b_2 b_4}{2}\Big[(x_2^2 + y_2^2)^2(x_3x_4 + y_3y_4) + (x_3x_4 - y_3y_4)(x_2^4 + y_2^4 - 6x_2^2y_2^2) \\ & + 4x_2y_2(x_2^2 - y_2^2)(x_3y_4 + y_3x_4)\Big]. \end{aligned} \tag{15}$$

The last two cases are non-trivial generalizations of Duistermaat integrable cases for the 1 : 1 : 2 resonance and seem to be new. Similarly as in above resonance, the integrals G_1, G_2 and G_3 do not come from some obvious symmetries.

We do not present the proof of this theorem, but we just mention the steps we take. Firstly, we use the Lyapounov's method, which we will explain here. Consider a holomorphic vector field X, defined on some complex analytic manifold.

$$\dot{\mathbf{z}} = X(\mathbf{z}). \tag{16}$$

Let $\psi(t)$ be a particular solution of (16). The variational equation along this solution is given by

$$\dot{\boldsymbol{\eta}} = X'(\psi(t))\boldsymbol{\eta}. \tag{17}$$

Then, Lyapounov, 1894 has made the following observation

If the linear system (17) *has a multi-valued solution, then the same holds for the non-linear system* (16)*.*

Summing up, if the general solution of the variational equation is not single-valued, then the non-linear system does not admit an additional analytic first integral.

Using this argument, we show that for a lot of parameter values the solutions of the variational equations along a simple solution of the canonical equations of (12) are not single-valued, hence the additional integrals do not exist.

Lastly, for the values of the parameters on the hypersurface $b_1 b_4 - b_2 b_3 = 0$ we utilize the Morales-Ramis theory, which is already well known to be repeated again J. J. Morales-Ruiz, 1999

Let us give one more example of possible application of above results in addition to those mentioned in the abstract, that is, coupled elastic pendulums or coupled double pendulums, both in 1 : 2 resonance.

Recall the generalized two-degrees of freedom Hénon-Heiles system which is defined by the Hamiltonian

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(A_1 x_1^2 + B_1 x_2^2) + \alpha_1 x_1^2 x_2 + \frac{\beta_1}{3} x_2^3, \qquad (18)$$

where (x_1, x_2, y_1, y_2) are canonical coordinates in $\mathbb{R}^4$. The original Hénon-Heiles Hamiltonian agrees with (18) when $A_1 = B_1 = \alpha_1 = 1, \beta_1 = -1$. This system is a paradigmatic model in celestial, statistical and quantum mechanics. The integrability is studied in J. J. Morales-Ruiz, 1999 with the help of Morales-Ramis theory. Furthermore, this model is extended to three or more degrees of freedom systems.

Consider the following generalization

$$\begin{aligned} H &= \frac{1}{2}(y_1^2 + y_2^2 + y_3^2 + y_4^2) + \frac{1}{2}(x_1^2 + x_2^2) + \frac{4}{2}(x_3^2 + x_4^2) \\ &+ \alpha_1 x_1^2 x_2 + \frac{\beta_1}{3} x_2^3 + \alpha_2 x_3^2 x_4 + \frac{\beta_2}{3} x_4^3 + R(x_1, \ldots, x_4, y_1, \ldots, y_4). \end{aligned} \tag{19}$$

This Hamiltonian is clearly in 1 : 1 : 2 : 2 resonance. Here R represent the coupling terms, containing necessarily resonant monomials of third order. The integrability of (19) is yet to be investigated, but our results can be applied when studying the truncated to order three normal form.

2. Simplification of the normal form

Now, we simplify the normal form (1) in order to make the integrability analysis easier.

First, we get rid of the quadratic part with the help of non-autonomous canonical transformation, see Christov, 2012. In this way, we obtain

$$\begin{aligned} \overline{H} = \overline{H}_3 = a_1[x_3(x_1^2 - y_1^2) + 2x_1y_1y_3] + a_3[x_4(x_1^2 - y_1^2) + 2x_1y_1y_4] \\ + a_5[x_3(x_2^2 - y_2^2) + 2x_2y_2y_3] + a_7[x_4(x_2^2 - y_2^2) + 2x_2y_2y_4] \\ + a_9[x_3(x_1x_2 - y_1y_2) + y_3(y_1x_2 + x_1y_2)] \\ + a_{11}[x_4(x_1x_2 - y_1y_2) + y_4(y_1x_2 + x_1y_2)]. \end{aligned} \tag{20}$$

Next, we want to adapt the Duistermaat's approach, used for the simplification of the 1 : 1 : 2 normal form to our case. We want to perform a symplectic transformation on (x_1, y_1, x_2, y_2)-space with (x_3, y_3) and (x_4, y_4) fixed, which leaves H_2 invariant and as a result to achieve a normal form (20) with $a_9 = a_{11} = 0$.

Introduce the complex coordinates

$$z_j = x_j + iy_j, \qquad \zeta_j = x_j - iy_j, \quad j = 1, \ldots, 4.$$

In these coordinates $\overline{H}$ becomes

$$\begin{aligned}\overline{H} = \frac{1}{2}z_3(a_1\zeta_1^2 + a_5\zeta_2^2 + 2a_9\zeta_1\zeta_2) + \frac{1}{2}z_4(a_3\zeta_1^2 + a_7\zeta_2^2 + 2a_{11}\zeta_1\zeta_2) \\ + \frac{1}{2}\zeta_3(a_1z_1^2 + a_5z_2^2 + 2a_9z_1z_2) + \frac{1}{2}\zeta_4(a_3z_1^2 + a_7z_2^2 + 2a_{11}z_1z_2).\end{aligned} \tag{21}$$

Consider the two quadratic forms

$$Z_1(z_1, z_2) = a_1z_1^2 + a_5z_2^2 + 2a_9z_1z_2, \qquad Z_2(z_1, z_2) = a_3z_1^2 + a_7z_2^2 + 2a_{11}z_1z_2 \tag{22}$$

and the corresponding matrices

$$S_1 = \begin{pmatrix} a_1 & a_9 \\ a_9 & a_5 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} a_3 & a_{11} \\ a_{11} & a_7 \end{pmatrix}. \tag{23}$$

Recall that the standard Hermitian product in $\mathbb{C}^2 = (z_1, z_2)$ - space, $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ has as its real part the usual inner product and as its imaginary part the symplectic form in the (x_1, y_1, x_2, y_2) - space. Since the unitary transformations preserve the standard Hermitian product, we can reformulate our problem as follows: keeping $(z_3, \zeta_3), (z_4, \zeta_4)$ fixed we look for a unitary transformation on $\mathbb{C}^2 = (z_1, z_2)$ - space, which simultaneously diagonalize S_1 and S_2. The quadratic part $H_2 = \frac{1}{2}z_1\zeta_1 + \frac{1}{2}z_2\zeta_2 + z_3\zeta_3 + z_4\zeta_4$ is automatically preserved. In this way we get a simplification of the quadratic forms (22), and hence, a simplification of the normal form (21). Observe that for the simplification of the 1 : 1 : 2 normal form one complex symmetric matrix has been diagonalized. Here we look for a unitary transformation which simultaneously diagonalizes two matrices and that naturally comes with certain cost.

The problem of simultaneous diagonalization of complex Hermitian or symmetric matrices has a long history, see for instance Bustamante et al, 2020 and the book cited below.

Since the matrices S_1 and S_2 come from quadratic forms, we need their simultaneous diagonalization by congruence. Luckily, there exists such kind a result which fits our purposes. We formulate that part which concerns our study.

Theorem (Horn, Johnson, Matrix Analysis, Cambridge University Press, 2013, Theorem 4.5.15, page 286)

Let $A, B \in M_n$ be given (M_n denotes the set of $n \times n$ matrices with complex entries).

(a) Suppose that A and B are Hermitian. There is a unitary $U \in M_n$ and real diagonal real diagonal Λ, M such that

$$A = U\Lambda U^*, \qquad B = UMU^*$$

if and only if AB is Hermitian, that is, $AB = BA$.

(b) Suppose A and B are symmetric. There is a unitary $U \in M_n$ and real diagonal Λ, M such that

$$A = U\Lambda U^T, \qquad B = UMU^T$$

if and only if $A\overline{B}$ is Hermitian, that is, $A\overline{B} = B\overline{A}$.

Due to the considered normal form (20) the matrices S_1 and S_2 are real symmetric. They commute when

$$a_1a_{11} - a_3a_9 + a_9a_7 - a_{11}a_5 = 0. \tag{24}$$

Then, according to the above result, there exist a unitary matrix U such that

$$S_1 = U\Lambda U^T, \qquad S_2 = UMU^T \tag{25}$$

with

$$\Lambda = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \qquad M = \begin{pmatrix} b_3 & 0 \\ 0 & b_4 \end{pmatrix}, \qquad b_j \in \mathbb{R}.$$

Moreover, we can additionally act on both Λ, M by a unitary matrix U_1 of the form $U_1 = \mathrm{diag}(e^{i\psi}, e^{i\phi})$ $\Lambda_1 = U_1 \Lambda U_1^T$, $M_1 = U_1 M U_1^T$ to achieve that the entries of Λ_1 are non-negative. Hence,

$$\Lambda_1 = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad b_1 \geq b_2 \geq 0, \quad M_1 = \begin{pmatrix} b_3 & 0 \\ 0 & b_4 \end{pmatrix}, \quad b_3, b_4 \in \mathbb{R} \tag{26}$$

and the transformed normal form (20) reads

$$\overline{H} = \frac{1}{2}\Big[b_1(z_3\zeta_1^2 + \zeta_3 z_1^2) + b_2(z_3\zeta_2^2 + \zeta_3 z_2^2) + b_3(z_4\zeta_1^2 + \zeta_4 z_1^2) + b_4(z_4\zeta_2^2 + \zeta_4 z_2^2)\Big]. \tag{27}$$

Returning to cartesian coordinates we get (12).

3. Concluding remarks

In this work, we consider the integrability of the truncated to order three normal form of the two fold 1 : 2 resonance. We adopt the special case of the normal from, studied in Mazrooei-Sebdani et al, 2021 for the main object of our investigation with a little rearrangement to gain 1 : 1 : 2 : 2 resonance. This is done in order to use the knowledge of the integrable cases in 1 : 2 : 2 and 1 : 1 : 2 resonance. Indeed, we deduce from last mentioned resonances new integrable cases and in this way enlarge the list given in the above paper.

Then, we further simplify the normal form and prove a non-integrability theorem. We prove this theorem by analyzing only the first variational equations along certain particular solutions of the Hamiltonian system under consideration. First, we apply the Lyapounov's argument and show that for a lot of the parameter values the solutions of the variational equations along a simple particular solution are not single-valued. That prevents the existence of additional first integrals.

For the values of parameters on the hypersurface $b_1b_4 - b_2b_3 = 0$ we make use of Morales-Ramis theory to prove non-integrability. Along the application of this theory, two exceptional case turned up: one of them more or less natural, the other one seem to be new and non-trivial, for which the full set of involutive first integrals exists.

Let us note that our study on the integrability of 1 : 1 : 2 : 2 resonance cannot in any way be regarded as conclusive. If the more general normal form is considered, other cases of integrability could also appear. Of course, this more general normal form contains more parameters and that makes the investigation more involved.

To summarize, the simplified normal form (12) is integrable only if:

- $b_1 = b_3 = 0$;
- $b_2 = b_4 = 0$;

- $b_1 = b_2 = 0$;
- $b_3 = b_4 = 0$;
- $b_1 = b_2$ and $b_3 = b_4$;
- $b_1 = 2b_2$ and $b_3 = 2b_4$.

As a further task, it is definitely worth exploring the local and global properties of the geometry and the dynamics of all integrable cases, listed above: symmetries, reductions, action-angle coordinates, KAM-conditions, etc.

As we mentioned in the Introduction, there is no systematic study on Hamiltonian resonances with four and more degrees of freedom. In order to obtain some knowledge in that direction based on our study so far, it is natural to ask whether one can find any integrable cases in the normal form of 1 : 1 : 1

: 2 : 2 : 2 resonance, or more generally in

$$\underbrace{1 : \ldots : 1}_{m} : \underbrace{2 : \ldots : 2}_{m}$$

resonance. Since many models in physics, in particular in optics, molecular thermal vibrations, etc can be seen as resonant oscillatory systems, such a study makes sense. After all that would probably cast the light on the understanding why the integrability of the resonance 1 : 2 : . . . : 2 is exceptional.

Thank you
for your attention!