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SPECIAL FUNCTIONS OF FRACTIONAL CALCULUS AS SOLUTIONS OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

Virginia Kiryakova

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences, Sofia

Recently, there is an increasing interest and use of differential equations and systems of fractional order (that is, arbitrary one, not necessarily integer). The first ideas of the Fractional Calculus (FC) are dated yet in the end of 17th century, and are related to the name of Leibnitz, Newton, Riemann, Liouville, Euler etc. For long time this theory it was considered as an exotic variant of the classical Calculus, but there were challenging open problems about the possibilities of physical and geometrical interpretation of the integral and derivatives of fractional order (FO). Almost 3 centuries later, FC became an unavoidable tool for describing the evolution of various real systems with the help of fractional evolution equations: differential equations where the integer order derivatives in time or/and in space are replaced by operators of FC. Including non-local operators and allowing to take in mind the memory, they provide better mathematical models for various physical, engineering, control theory, biological and biomedical, chemical, earth, economics, etc. phenomena. In particular, to mention the anomalous diffusion, Brownian motion, stochastic, financial markets - as the Black-Scholes option, etc.

The development in theoretical and applied science has always required a knowledge of the properties of the Special Functions (SF), from the elementary trigonometric functions and error functions to the variety of SF of Mathematical Physics (named SF) as these of Bessel, Gauss, Tricomi, Laguerre, Struve, Airy, etc., appearing in studies of natural and social phenomena, in modeling of engineering problems, and numerical simulations. These "Classical SF" appear as solutions of differential and integral equations of *integer order*, mainly of 2nd order, but also of higher (integer) ones.

With the increasing role of the FC and the recognition that the fractional order models can describe better the fractal nature or the world, the solutions of the *fractional order differential and integral equations and systems* gained their important place and became unavoidable tools. These are the so-called Special Functions of Fractional Calculus (SF of FC), in the general case are Fox H-functions. Among them the basic role have the generalized Wright hypergeometric functions ${}_p\Psi_q(z)$ and in particular, the classical Mittag-Leffler (M-L) function $E_{\alpha,\beta}(z)$ and its various extensions (with multi-indices).

The classical FC is based on several (equivalent or alternative) definitions for the operators of integration and differentiation of arbitrary (including real fractional or complex) order, as continuation of the classical integration and differentiation of integer order $n \in \mathbb{N}$: the n-fold integration

$$R^n f(t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-2}} d\tau_{n-1} \int_0^{\tau_{n-1}} f(\tau_n) d\tau_n$$
$$= \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \qquad (1)$$

and the n-th order derivative $D^n f(t) = f^{(n)}(t)$.

The *Riemann-Liouville (R-L) integration of arbitrary order* $\delta > 0$ is defined by analogy, replacing $(n-1)!$ by $\Gamma(\delta)$:

$$R^\delta f(t) = D^{-\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} f(\tau) d\tau = t^\delta \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(t\sigma) d\sigma, \qquad (2)$$

while $R^0 f(t) \equiv f(t)$ is the identity operator, and the *semigroup property* is satisfied:

$$R^{\delta_1} R^{\delta_2} = R^{\delta_2} R^{\delta_1} = R^{\delta_1+\delta_2}, \quad \text{for } \delta_1 \geq 0, \delta_2 \geq 0.$$

A more general operator of fractional integration is the Erdélyi-Kober integral (E-K), allowing wider applications due to additional 2 parameters (γ and β):

$$I_\beta^{\gamma,\delta} f(t) = t^{-\beta(\gamma+\delta)} \int_0^t \frac{(t^\beta - \tau^\beta)^{\delta-1}}{\Gamma(\delta)} \tau^{\beta\gamma} f(\tau)\, d(\tau^\beta) \qquad (3)$$

$$= \int_0^1 \frac{(1-\sigma)^{\delta-1}\, \sigma^\gamma}{\Gamma(\delta)} f(t\sigma^{1/\beta})\, d\sigma, \quad \gamma \in \mathbb{R},\ \beta > 0.$$

The E-K operator is used essentially in our works on generalized FC. Namely, we consider *compositions of commutable E-K operators* but written *in form of single integrals involving special functions*, instead of by repeated integrals. We call them generalized fractional integrals (multiple E-K integrals), then introduce also the corresponding generalized fractional derivative of R-L and C-type.

The above definitions concern integrations of *nonnegative orders*, but cannot be used directly for the inverse operations, the differentiation as $D^\delta f(t) := R^{-\delta} f(t)$, $-\delta < 0$, the E-K FD, etc. However, a little trick is helpful for a suitable interpretation *to avoid divergent integrals*.

For noninteger $\delta > 0$ we take $n := [\delta] + 1$ (the smallest integer greater than δ), then by means of compositions of differentiation of order n and integration of nonnegative order $n - \delta \geq 0$, we can define properly two kinds of fractional order derivatives:

– the *R-L fractional derivative* by the differ-integral expression

$$D^\delta f(t) := D^n \, D^{\delta-n} f(t) = \left(\frac{d}{dt}\right)^n R^{n-\delta} f(t)$$
$$= \left(\frac{d}{dt}\right)^n \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^t (t-\tau)^{n-\delta-1} f(\tau) d\tau \right\}, \quad (4)$$

and

– the *Caputo* (Djrbashjan) *fractional derivative* by means of the integro-differential expression

$${}^*D^\delta f(t) := D^{\delta-n} \, D^n \, f(t) = R^{n-\delta} f^{(n)}(t)$$
$$= \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^t (t-\tau)^{n-\delta-1} f^{(n)}(\tau) d\tau \right\}. \quad (5)$$

In suitable functional spaces (continuous, L-integrable, analytical),

$$D^\delta R^\delta f(t) = {}^*D^\delta R^\delta f(t) = f(t).$$

From the formula

$$D^{\delta}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\delta)}\, t^{\alpha-\delta}, \quad \delta > 0,\ \alpha > -1,$$

it follows an interesting situation, going in *conflict* with the classical Calculus: for $\alpha = 0$ the R-L fractional derivative of a constant is zero *only* for positive integer values $\delta = n = 1, 2, 3, \dots$, but not for arbitrary $\delta \notin \mathbb{N}_+$:

$$D^{\delta}\{c\} = c\, \frac{t^{-\delta}}{\Gamma(1-\delta)}, \quad \text{while Caputo's der.: } {}^{*}D^{\delta}\{c\} = 0, \text{ always.}$$

The R-L definition is preferred for the theoretical developments and their applications in pure mathematics, but the Caputo derivative is more suitable for mathematical models of real phenomena also for more important reasons: to be able to consider problems where the initial conditions are given by limit values of integer order derivatives at the lower terminal ($t = 0$), instead of fractional order integrals or derivatives, that can hardly be interpreted physically.

In the case of E-K derivative and generalized fractional derivatives, the corresponding definitions are developed (Kiryakova) by suitable (more complicated) expressions for both

Let us consider some few simplest *examples of fractional order (ODE) differential equations* related to models from mechanics, as ultraslow and intermediate processes, and diffusion-wave phenomena. The fractional differential equations of order $\alpha > 0$:

$$\frac{d^\alpha u(t)}{dt^\alpha} + u(t) = 0, \quad t > 0$$

with initial value conditions of the form $u^{(k)}(+0) = c_k,\ k = 0, 1, 2, ..., n, \ \ n \in \mathbb{N}, n-1 < \alpha \leq n$, are usually referred to as the fractional relaxation equations (if $0 < \alpha \leq 1$), or fractional oscillation equations (if $1 < \alpha \leq 2$) equations. For integer orders α the IVPs for the above equation can be solved by elementary functions:

– if $\alpha = 1$: eq. of relaxation, $u(t) = c_0 \exp(-t)$;

– if $\alpha = 2$: eq. of oscillation, $u(t) = c_0 \cos t + c_1 \sin t$.

But in the fractional order cases, the so-called Mittag-Leffer function $E_{\alpha,\beta}(t)$ has a key role, and has been titled by Mainardi as "Queen function of FC".

One of the *(PDE) partial differential equations of fractional order* is well popular, as obtained from the classical diffusion or wave equations by replacing the first–, or resp. the second–order time derivative by a fractional derivative of order α with $0 < \alpha < 2$. It has the form

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \lambda^2 \frac{\partial^2}{\partial x^2} u(x,t),$$

where x, t denote the space-time variables (in one-dimensional space) and $u(x,t)$ is the response field variable. Equations of this form have been introduced in physics, with $0 < \alpha < 1$ by Nigmatullin to describe the diffusion process in media with fractal geometry; and for $1 < \alpha < 2$ by Mainardi to describe the propagation of mechanical diffusive waves in viscoelastic media which exhibit a power law creep. These types of equations, called as fractional diffusion-wave equations, have been treated by several authors by means of different approaches (either by Mellin transforms techniques and solutions in terms of H-functions of Fox, or by Laplace transform allowing to obtain for the fundamental solution of the IVP via the Green function), but all of them finally lead to use of the SF of FC as for explicit solutions.

In Financial Mathematics, the so called Black-Scholes (B-S) model in European options is popular, and extensively studied now by FO modifications. The original classical order B-S model for the value of an option is described by the heat-type PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-\tau)S\frac{\partial V}{\partial S} - rV = 0, \quad (S,t) \in \mathbb{R}^+ \times (0,T),$$

where ... But recently different FO variants of the B-S equation are considered, for example with a FO time-derivative:

$$\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-\tau)S\frac{\partial V}{\partial S} - rV = 0, \quad \text{with} \quad 0 < \alpha \leq 1.$$

Different methods are used, for example, reconstruction of variational iteration method, to find that the Mittag-Leffler function $E_\alpha = E_{\alpha,1}$ appears in the solution of the fractional B-S eq.

$$\frac{\partial^\alpha V}{\partial t^\alpha} = \frac{\partial^2 V}{\partial S^2} + (k-1)\frac{\partial V}{\partial S} - kV = 0, \quad V(S,0) = \max\{e^S - 1, 0\},$$

as:

$$V(S,t) = \max\{\, e^S - 1, 0\}\, E_\alpha(-kt^\alpha) + \max\{e^S, 0\}\,(1 - E_\alpha(-kT^\alpha)).$$

For $\alpha = 1$, the exact solution of the classical B-S eq. includes the exponential function, $\exp(-kt) = E_{1,1}(-kt)$.

The different effects of *use of either R-L or Caputo-type fractional derivatives* is seen from the simplest example of the following Cauchy problems. The Cauchy problem for the FO differential equation *with R-L derivative*, of the form

$$\begin{cases} D^{\delta}y(t) - \lambda y(t) = f(t)\ , \quad t > 0, \\ D^{\delta-j}y(t)|_{t=0} = b_j\ ,\ j = 1, 2, ..., n, \end{cases} \qquad n-1 < \delta \le n,$$

has its solution in terms of Mittag-Leffler function (Examples 42.1, 42.2 in Samko-Kilbas-Marichev book):

$$y(t) = |!\sum_{j=1}^{n} b_j\ t^{\delta-j} E_{\delta,1+\delta-j}(\lambda t^{\delta}) + \int_0^t (t-\tau)^{\delta-1}\ E_{\delta,\delta}[\lambda(t-\tau)^{\delta}]\ f(\tau)d\tau.$$

And for the same differential equation *with Caputo fractional derivative*, the IVP stated in more comprehensive form

$$\begin{cases} {}^{*}D^{\delta}y(t) - \lambda y(t) = f(t)\ , \quad t > 0, \\ y^{i-1}(0) = b_i\ ,\ i = 1, 2, ..., n, \end{cases} \qquad n-1 < \delta \le n,$$

has solution, again in term of M-L functions, have same structure (Podlubny; Srivastava-Kilbas-Trujillo) but with different elements

$$y(t) = \sum_{i=1}^{n} c_i\ t^{i-1} E_{\delta,i}(\lambda t^{\delta}) + \int_0^t (t-\tau)^{\delta-1}\ E_{\delta,\delta}[\lambda(t-\tau)^{\delta}]\ f(\tau)d\tau.$$

The Bateman Project, planned as a "Guide to the Functions": During his last years, Harry Bateman worked on a project whose successful completion, as he believed, would be of great value to scientists in all fields. He planned an extensive compilation of "special functions" – solutions of a wide class of mathematically and physically relevant functional equations. He intended to tabulate properties of such functions, their inter-relations between, representations in various forms, and to construct tables of important definite integrals involving such functions ... Thus, to collect as easily extracted information, the large amount of scattered research on special functions. In the time of Bateman's death (1946) his notes amounted to a card-catalogue of several dozen cardboard boxes (the famous "shoe-boxes"). Bateman planned his Project as a "Guide to the Functions". It resulted in publication of 5 important volumes, edited by A. Erdélyi, with W. Magnus, F. Oberhettinger, F.G. Tricomi: "*Higher Transcendental Functions*" (1953-1955) and "*Tables of Integral Transforms*" (1954). This inspired the author to modify the Bateman title as a title of the following survey paper ("Mathematics", OA, 2021).

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Review

A Guide to Special Functions in Fractional Calculus

Virginia Kiryakova

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria; virginia@diogenes.bg

Abstract: *Dedicated to the memory of Professor Richard Askey (1933–2019) and to pay tribute to the Bateman Project*. Harry Bateman planned his "shoe-boxes" project (accomplished after his death as *Higher Transcendental Functions*, Vols. 1–3, 1953–1955, under the editorship by A. Erdélyi) as a "*Guide to the Functions*". This inspired the author to use the modified title of the present survey. Most of the standard (classical) Special Functions are representable in terms of the Meijer G-function and, specially, of the generalized hypergeometric functions ${}_pF_q$. These appeared as solutions of differential equations in mathematical physics and other applied sciences that are of integer order, usually of second order. However, recently, mathematical models of fractional order are preferred because they reflect more adequately the nature and various social events, and these needs attracted attention to *"new" classes* of special functions as their solutions, the so-called *Special Functions of Fractional Calculus (SF of FC)*. Generally, under this notion, we have in mind the Fox H-functions, their most widely used cases of the Wright generalized hypergeometric functions ${}_p\Psi_q$ and, in particular, the Mittag–Leffler type functions, among them the "Queen function of fractional calculus", the Mittag–Leffler function. These fractional indices/parameters extensions of the classical special functions became an unavoidable tool when fractalized models of phenomena and events are treated. Here, we try to review some of the basic results on the theory of the SF of FC, obtained in the author's works for more than 30 years, and support the wide spreading and important role of these functions by several examples.

Keywords: special functions; generalized hypergeometric functions; fractional calculus operators; integral transforms

MSC: 33C60; 33E12; 26A33; 44A20

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1. Historical Introduction

Special functions are particular mathematical functions that have more or less established names and notations due to their importance in mathematical analysis, functional analysis, geometry, physics, astronomy, statistics or other applications (Wikipedia: Special Functions [1]). It might be Euler, who started to talk, since 1720, about lots of the standard special functions. He defined the Gamma-function as a continuation of the factorial, also the Bessel functions and looked after the elliptic functions. Several (theoretical and applied) scientists started to use such functions, introduced their notations and named them after famous contributors. Thus, the notions as the Bessel and cylindrical functions; the Gauss, Kummer, Tricomi, confluent and generalized hypergeometric functions; the classical orthogonal polynomials (as Laguerre, Jacobi, Gegenbauer, Legendre, Tchebisheff, Hermite, etc.); the incomplete Gamma- and Beta-functions; and the Error functions, the Airy, Whittaker, etc. functions appeared and a long list of handbooks on the so-called "*Special Functions of Mathematical Physics*" or "*Named Functions*" (we call them also "*Classical Special Functions*") were published. We mention only some of them in this survey.

As Richard Askey (*to whose memory we dedicate this survey*) confessed in his lectures [2] on orthogonal polynomials and special functions: "Now, there are relatively large num-

A historical note from *Bateman Project* (HTF, Vol.1):

"... Of all integrals which contain Gamma functions in their integrands the most important ones are the so-called *Mellin-Barnes integrals*. Such integrals were first introduced by S. Pincherle, in 1888; their theory has been developed in 1910 by H. Mellin, . . . and they were used for a complete integration of the hypergeometric differential equation by E.W. Barnes, 1908."

Definition (Ch. Fox, 1961). The Fox H-function is a generalized hypergeometric function, defined by means of the Mellin-Barnes type contour integral

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{array}{l}(a_i,A_i)_1^p\\(b_j,B_j)_1^q\end{array}\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\mathcal{H}_{p,q}^{m,n}(s)\,z^{-s}ds,\quad \text{with}$$

$$\mathcal{H}_{p,q}^{m,n}(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{i=1}^{n}\Gamma(1-a_i-A_is)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{i=n+1}^{p}\Gamma(a_i+A_is)},\quad z\neq 0, \qquad (6)$$

Note that the integrand $\mathcal{H}_{p,q}^{m,n}(s)$ with $s\mapsto -s$ is practically the Mellin transform of the H-function (6).

The orders (m, n, p, q) are non negative integers: $0 \leq m \leq q$, $0 \leq n \leq p$. The parameters $A_i > 0, B_j > 0$ are positive, and a_i, b_j, $i = 1, \ldots, p; j = 1, \ldots, q$ can be arbitrary complex such that $A_i(b_j{+}l) \neq B_j(a_i{-}l'{-}1)$, $l, l' = 0, 1, 2, \ldots; i = 1, \ldots, n; j = 1, \ldots, m$. Here $\mathcal{L}$ is a suitable contour (of 3 possible types in $\mathbb{C}$: $\mathcal{L}_{-\infty}$, $\mathcal{L}_{\infty}$, $(\gamma - i\infty, \gamma + i\infty)$) that needs to separate the poles of the two groups of Γ-functions in the numerator of $\mathcal{H}_{p,q}^{m,n}(s)$, none of them coinciding for the assumed conditions:

The details on the properties of the Fox H-function can be found in many contemporary handbooks on SF (and FC).
Note that the *H-function is an analytic function of z* in circle domains $|z| < \rho$ (or in sectors of them, or in the whole $\mathbb{C}$), depending on the above parameters and the contours $\mathcal{L}$.

See for example, books of Prudnikov-Brychkov-Marichev (Vol.3), Kilbas and Saigo, Mathai-Saxena, etc. For studies on the behavior of the H-function around the singular points ($z = 0$, $|z| = \rho$, $z = \infty$), one can see also Karp (2020), commenting and revisiting the Braaksma' s results (1962-1964).

Meijer's G-function (C.S. Meijer, 1936-1941):

If all $A_i = B_j = 1,\ i = 1, ..., p; j = 1, ..., q$, the H-function $H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}(a_i, 1)_1^p \\ (b_j, 1)_1^q\end{matrix}\right]$ reduces to the *Meijer G-function*

$$G^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}(a_i)_1^p \\ (b_j)_1^q\end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{G}^{m,n}_{p,q}(s)\, z^{-s} ds$$

$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + s) \prod_{i=1}^{n} \Gamma(1 - a_i - s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - s) \prod_{i=n+1}^{p} \Gamma(a_i + s)}\, z^{-s} ds, \quad z \neq 0. \tag{7}$$

In this case, the behavior of (7) depends on the conditions simplified as: $\rho = 1$, $\Delta = q - p$, $\delta = m + n - \frac{p+q}{2}$.

Although simpler than the H-function, the G-function is yet enough general as it incorporates most of the Classical SF (Named SF) and many elementary functions. Extensive lists of examples and operational properties of the G-functions can be found even in the Bateman project (HTF), Vol.1, Ch.5; and in recent SF books.

Now, we attract attention to the most typical examples of SF of FC which are Fox H-functions but are *not* reducible to Meijer G-functions in the general case (of *irrational* A_j, B_k).

Generalized Wright Hypergeometric Functions:

The *Wright generalized hypergeometric function* ${}_p\Psi_q(z)$, called also *Fox-Wright function* (F-W g.h.f.) is defined as:

$$ {}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{array}\bigg| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)\ldots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\ldots\Gamma(b_q+kB_q)}\frac{z^k}{k!} \tag{8}$$

$$ = H^{1,p}_{p,q+1}\left[-z\,\bigg|\,\begin{array}{c}(1-a_1,A_1),\ldots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q)\end{array}\right], \tag{9}$$

in terms of the more general Fox H-function. Denote

$$\rho=\prod_{i=1}^{p}A_i^{-A_i}\prod_{j=1}^{q}B_j^{B_j},\quad \Delta=\sum_{k=1}^{j}B_j-\sum_{i=1}^{p}A_i.$$

If $\Delta>-1$, the ${}_p\Psi_q$-function is an entire function of z, $z\in\mathbb{C}$; and if $\Delta=-1$, this series is absolutely convergent in the disk $\{|z|<\rho\}$, while for $|z|=\rho$ if $\Re(\mu)=\Re\left\{\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i+\frac{p-q}{2}\right\}>1/2.$

For $A_1 = \cdots = A_p = 1, B_1 = \cdots = B_q = 1$ in (8) and (9), the Wright g.h.f. reduces to the more popular (classical) *generalized hypergeometric ${}_pF_q$-function*, which is also a Meijer's G-function:

$$
\begin{aligned}
{}_p\Psi_q \left[\begin{array}{c} (a_1, 1), \ldots, (a_p, 1) \\ (b_1, 1), \ldots, (b_q, 1) \end{array} \middle| z \right] &= c\; {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q;\; z) \\
&= c \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!} \qquad (10) \\
&= c\; G^{1,p}_{p,q+1} \left[-z \middle| \begin{array}{c} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_q \end{array} \right];
\end{aligned}
$$

$$
\text{with} \quad c = \left[\prod_{i=1}^{p} \Gamma(a_i) \Big/ \prod_{j=1}^{q} \Gamma(b_j) \right], \; (a)_k := \Gamma(a+k)/\Gamma(a).
$$

With except for some integer values of parameters (when the series breaks to a polynomial), ${}_pF_q$ is convergent for all finite z if $p \le q$; for $|z| < 1$ if $p = q+1$; and diverges for all $z \neq 0$ if $p > q+1$.

Most of the classical SF (and elementary functions) appear as particular cases of ${}_pF_q$-function.

The Queen function of FC:

The Mittag-Leffler (M-L) functions E_α (Mittag-Leffler, 1902-1905), resp. the more general one $E_{\alpha,\beta}(z)$ (Wiman 1905, Agarwal, 1953), studied later also by Dzrbashjan (1954, 1960):

$$E_{\alpha,\beta}(z) = \sum_0^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad E_{\alpha,1}(z) := E_\alpha(z), \quad \alpha > 0, \beta > 0 \quad (11)$$

have been almost ignored, for a long time, in the handbooks on SF and in existing tables of Laplace transforms. These "*fractional exponential functions*" are natural extensions of the exp-function and trigonometric functions ($\alpha = 1$, $\alpha = 2$), for example of

$$y_1(z) = E_1(z) = \exp(z) \quad \text{and} \quad y_2(z) = E_2(-z^2) = \cos z$$

which satisfy integer (1st and 2nd) order differential equations as

$$D^1 y_1(\lambda z) = \lambda y_1(\lambda z), \quad D^2 y_2(z) = -\lambda^2 y_2(\lambda z).$$

However, the "true" M-L functions are solutions of fractional order DEs, as for example, the α-exponential (Rabotnov) function:

$$D^\alpha y(\lambda z) = \lambda y(\lambda z) \quad \text{with} \quad y(z) = z^{\alpha-1} E_{\alpha,\alpha}(z^\alpha).$$

The *Laplace transform* images for the M-L type functions and their k-th derivatives are (Podlubny):

$$\mathcal{L}\left\{z^{\alpha k+\beta-1}\, E^{(k)}_{\alpha,\beta}(\pm\lambda z^{\alpha})) ;\, s\right\} = \frac{k!\, s^{\alpha-\beta}}{(s^{\alpha} \mp \lambda)}^{k+1}, \quad \Re s > |\lambda|^{1/\alpha}.$$

A M-L type function with 3 parameters, known as the *Prabhakar function* (T.R. Prabhakar, 1971) is also often studied and used in FC, for $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}\, \alpha > 0$ and with the Pochhammer symbol $(\gamma)_0 = 1, (\gamma)_k = \Gamma(\gamma + k)/\Gamma(\gamma)$:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} \text{ with } \mathcal{L}\left\{E^{\gamma}_{\alpha,\beta}(\lambda z^{\alpha}); s\right\} = \frac{s^{-\beta}}{(1 - \lambda s^{-\alpha})^{\gamma}}. \tag{12}$$

For $\gamma = 1$ we get the M-L function $E_{\alpha,\beta}$, and if aslo $\beta = 1$, it is E_{α}.

These M-L type functions are *simple cases of the Wright g.h.f. and of the H-function*, namely:

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left[\begin{array}{c} (1,1) \\ (\beta,\alpha) \end{array}\middle| z\right] = H^{1,1}_{1,2}\left[-z\middle| \begin{array}{c} (0,1) \\ (0,1),(1-\beta,\alpha) \end{array}\right],$$

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} {}_1\Psi_1\left[\begin{array}{c} (\gamma,1) \\ (\beta,\alpha) \end{array}\middle| z\right] = H^{1,1}_{1,2}\left[-z\middle| \begin{array}{c} (1-\gamma,1) \\ (0,1),(1-\beta,\alpha) \end{array}\right].$$

Another generalization of the M-L function (11) with additional parameters, $l \in \mathbb{C}, \mu \in \mathbb{R}$, was introduced and studied by *Gorenflo-Kilbas-Rogosin* (1998) as a solution of Abel-Volterra FO integral equation, and recently is often exploited in the solutions of multi-term fractional differential equations:

$$E_{\alpha,\mu,l}(z) = \sum_{k=0}^{\infty} c_k z^k, \quad \text{with} \quad c_k = \prod_{j=0}^{k-1} \frac{\Gamma[\alpha(j\mu + l) + 1]}{\Gamma(\alpha(j\mu + l + 1) + 1]}.$$

It is interesting to note also the so-called *generalized Le Roy type functions*, $\delta > 0$ (extending Le Roy f. with $\alpha = \beta = 1 \to [k!]^{\delta}$):

$$^{(\delta)}F_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k + \beta)]^{\delta}}, \quad ^{(\delta)}F_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{[\Gamma(\alpha k + \beta)]^{\delta}} \frac{z^k}{k!},$$

studied recently by Gerhold, Garra, Rogosin, Mainardi, Garrappa, Pogany, Tomovski, etc. Similarly to the other M-L type functions, these are proved to be entire functions (for $\Re(\alpha) > 0$, $\beta, \gamma \in \mathbb{C}$, $\delta > 0$). These are involved in solutions of various FO problems; say in construction of a Conway–Maxwell–Poisson distribution (important due to its ability to model count data with different degrees of over- and under-dispersion), in stochastic models, etc.

Extensions of the classical M-L functions (11), Kiryakova (since 1996-1999, etc.), the 2 parameters α and $\beta \longrightarrow$ 2 sets (vector indices, $2m$ indices): $(\alpha_1, ..., \alpha_m)$ and $(\beta_1, ..., \beta_m)$, $\alpha_i > 0$, $m \geq 1$. These multi-index M-L functions include many of the SF of FC as particular cases (next slides!) and appear as solutions of fractional order problems of multi-order $(\alpha_i)_1^m$, $m \geq 1$:

$$E_{(\alpha_i),(\beta_i)}(z) := E^{(m)}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1) \ldots \Gamma(\alpha_m k + \beta_m)}$$

$$= {}_1\Psi_m\left[\begin{array}{c} (1,1)_1^m \\ (\beta_i, \alpha_i)_1^m \end{array} \middle| z \right] = H^{1,1}_{1,m+1}\left[-z \middle| \begin{array}{c} (0,1)_1^m \\ (0,1)_1^m, (1-\beta_i, \alpha_i)_1^m \end{array} \right]. \quad (13)$$

Later, Paneva-Konovska (2011): multi-index analogues also of the Prabhakar f., by $3m$ parameters, with $(\gamma_1, ..., \gamma_m)$ instead of γ:

$$E^{(\gamma_i),m}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k ... (\gamma_m)_k}{\Gamma(\alpha_1 k + \beta_1) ... \Gamma(\alpha_m k + \beta_m)} \frac{z^k}{(k!)^m}$$

$$= c \; {}_m\Psi_{2m-1}\left[\begin{array}{c} (\gamma_i, 1)_1^m \\ (\beta_i, \alpha_i)_1^m (1,1) ... (1,1) \end{array} \middle| z \right] = c \; H^{1,m}_{m,2m}(-z) \; .... \quad (14)$$

There: $(\gamma_i)_k = \Gamma(\gamma_i + k)/\Gamma(\gamma_i)$ - the Pochhammer symbol; $c = \left[\prod_{i=1}^{m} \Gamma(\gamma_i)\right]^{-1}$. Evidently, $\forall\ \gamma_i = 1,\ i = 1, ..., m$ lead to above $2m$ parameter M-L type function (13), while for $m = 1$ we get the Prabhakar function.

Note that later, Kilbas-Rogosin-Koroleva (2006, 2013) extended the $2m$ index M-L functions $E^{(m)}_{(\alpha_i),(\beta_i)}(z)$ with all $\alpha_i > 0$ to the case when all $\alpha_i, i = 1, ..., m$ can be real and different from zero!

Studies on multi-index (vector) M-L functions began by Luchko et al. (1995, 1999) where by means Operational Calculus tools he introduced the multi-variable variant ($\beta > 0, \alpha_i > 0$, integers $l_i \geq 0,\ i = 1, ..., m$):

$$E_{(\alpha_1,...,\alpha_m),\beta}(z_1, ..., z_m) = \sum_{k=0}^{\infty} \sum_{l_1+...+l_m=k} (k; l_1, ..., l_m) \frac{\prod_{i=1}^{m} z_i^{l_i}}{\Gamma(\beta + \sum_{i=1}^{m} \alpha_i l_i)},$$

with multinomial coeff. $(k; l_1, ..., l_m) = k!/(l_1)!...(l_m!)$. These multi-variable M-L type functions are useful to present solutions of multi-term FO problems in the recent works of many authors !

Basic theory of the multi-index M-L functions
(Kiryakova 1999–2010):

Theorem. *The multi-index M-L functions* (13) *are entire functions of order* ρ *with* $1/\rho = \alpha_1 + ... + \alpha_m$, *and type* σ*:*

$$1/\sigma = (\rho\alpha_1)^{\rho\alpha_1} \dots (\rho\alpha_m)^{\rho\alpha_m} > 1 \ \ (\textit{for } m > 1). \qquad (15)$$

Moreover, for each $\varepsilon > 0$ *we have an asymptotic estimate, as:*

$$|E_{(\alpha_i),(\beta_i)}(z)| \leq \exp\left((\sigma + \varepsilon)|z|^\rho\right), \ |z| \geq r_0 > 0,$$

with ρ, σ *as above,* $r_0(\varepsilon)$ *sufficiently large.*

We have shown also that the multi-index M-L functions are eigen-functions of the Gelfond-Leontiev (1951) operators of generalized differentiation that we generated by means of the coefficients of these entire functions. That is, $E_{(\alpha_i),(\beta_i)}(z)$ satisfies a differential equation of multi-order $(1/\alpha_1, ..., 1/\alpha_m)$:

$$D_{(\alpha_i),(\beta_i)}\, E_{(\alpha_i),(\beta_i)}(\lambda z) = \lambda\, E_{(\alpha_i),(\beta_i)}(\lambda z), \quad \lambda \neq 0,$$

where $D_{(\alpha_i),(\beta_i)}$ is a generalized fractional differentiation operator of the form $z^{-1}\, D_{(1/\alpha_i),m}^{(\beta_i - \alpha_i - 1),(\alpha_i)}$ (in sense of Kiryakova, 1994).

The $(3m)$-parameters M-L type functions (14) are also entire functions with the same order and type as in (15) (Paneva-Konovska, 2011-2106).

Other useful properties of the multi-M-L functions, aside from being examples of the Wright g.h.f. an H-functions, are shortly as:

Lemma. *The following Mellin-Barnes type integral representation holds:*

$$E_{(\alpha_i),(\beta_i)}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\Gamma(1-s)}{\prod_{i=1}^{m} \Gamma(\beta_i - s\alpha_i)} (-z)^{-s} ds, \quad z \neq 0,$$

based on the Mellin transform

$$\mathcal{M}\left\{E_{(\alpha_i),(\beta_i)}(-z); s\right\} = \frac{\Gamma(s)\Gamma(1-s)}{\prod_{i=1}^{m} \Gamma(\beta_i - s\alpha_i)}, \quad 0 < \Re(s) < 1. \quad (16)$$

As an analogue of the Laplace transform ($\mathcal{L}$) relationship between the classical M-L function (11) and the classical Wright function (that will be mentioned as particular case):

$$\mathcal{L}\left\{\varphi(\alpha,\beta;z);s\right\} = \frac{1}{s}E_{\alpha,\beta}(\frac{1}{s}),$$

we have derived the following *new relation*.

Lemma.

$$\mathcal{L}\left\{{}_0\Psi_m\left[\begin{array}{c} - \\ (\beta_1,\alpha_1),...,(\beta_m,\alpha_m) \end{array}\Bigg| z\right];s\right\} = \frac{1}{s}\,E_{(\alpha_i),(\beta_i)}(\frac{1}{s}),\ \ \Re(s)>0\,. \tag{17}$$

Note that we can consider the ${}_0\Psi_m$-functions on the left-hand side as *"fractional indices" analogues of the ${}_0F_m$-functions*, that is *of the hyper-Bessel functions $J^{(m)}_{\nu_1,\dots,\nu_m}$* of Delerue (1953), related to the hyper-Bessel operators as their eigenfunctions, as will be discussed further as special cases of (13).

The classical *Poisson integral formula*, representing the Bessel function via the cosine-function, can be written in terms of a fractional E-K fractional integral, as

$$J_\nu(z) = \frac{2}{\sqrt{\pi}\,\Gamma(\nu+1/2)} \left(\frac{z}{2}\right)^\nu \int_0^1 (1-t^2)^{\nu-1/2} \cos(zt)dt \\ = \frac{1}{\sqrt{\pi}} \left(\frac{z}{2}\right)^\nu \; I_{1/2}^{-1/2,\nu+1/2}\{\cos z\}. \quad (18)$$

Then, it has been extended (Kiryakova) for the *hyper-Bessel functions*, that is for the ${}_0F_{m-1}$-*functions*, and for the multi-index M-L functions, via generalized fractional integrals of the function $\cos_m$.

Other important analytical properties of the multi-index M-L functions can be found in series of our recent publications.

Many of the elementary and special functions are particular cases of $E_{\alpha,\beta}$, and much more – of the multi-index M-L functions $E_{(\alpha_i),(\beta_i)}(z)$

Examples of the M-L functions:

• $\alpha > 0$, $\beta = 1$: $E_{0,1}(z) = \dfrac{1}{1-z}$; $E_{1,1}(z) = \exp(z)$;
$E_{2,1}(z^2) = \cosh z$, $E_{2,1}(-z^2) = \cos z$; $E_{1/2,1}(z^{1/2}) =$
$\exp(z)\left[1 + \mathrm{erf}(z^{1/2})\right] = \exp(z)\,\mathrm{erfc}(-z^{1/2}) =$
$\exp(z)\left[1 + \dfrac{1}{\sqrt{\pi}}\gamma(\dfrac{1}{2}, z)\right]$ (the error functions, or incomplete gamma functions);

• $\beta \neq 1$: $E_{1,2}(z) = \dfrac{e^z - 1}{z}$; $E_{1/2,2}(z) = \dfrac{\mathrm{sh}\sqrt{z}}{z}$;
$E_{2,2}(z) = \dfrac{\mathrm{sh}\sqrt{z}}{\sqrt{z}}$; the Miller-Ross function $E_{1,\nu+1}(z)$, etc.

• $\beta = \alpha$: the α-exponential (Rabotnov) function
$y_\alpha(z) = z^{\alpha-1}E_{\alpha,\alpha}(\lambda z^\alpha)$, $\alpha > 0$.

One can prolong the list of cases of M-L functions ($m = 1$):

- The *trigonometric functions of order m*, and resp. the *hyperbolic functions of order m* (see Podlubny, 1999):

$$\cos_m(z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{mj}}{(mj)!} = E_{m,1}(-z^m), \text{ as the solution of IVP:}$$
$$y^{(m)}(z) = -y(z),\ y(0)=1, y^{(j)}(0)=0,\ j = 1, ..., m-1; \tag{19}$$

$$k_r(z,m) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{mj+r-1}}{(mj+r-1)!} = z^{r-1} E_{m,r}(-z^m),\ r = 1, 2, \ldots;$$

$$h_r(z,m) = \sum_{j=0}^{\infty} \frac{z^{mj+r-1}}{(mj+r-1)!} = z^{r-1} E_{m,r}(z^m),\ r = 1, 2, \ldots;$$

$$\cosh_m(z) = E_{m,1}(z^m)\ \ (r = 0).$$

Also, their *fractionalized versions*, as by Plotnikov (1979) and Tseytlin (1984):

$$Sc_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{(2-\alpha)m+1}}{\Gamma((2-\alpha)m+2)} = z\, E_{2-\alpha,2}(-z^{2-\alpha}),$$

$$Cs_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{(2-\alpha)m}}{\Gamma((2-\alpha)m+1)} = E_{2-\alpha,1}(-z^{2-\alpha}),$$

and by Luchko-Srivastava (1995), see also in Podlubny (1999):

$$\sin_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\Gamma(2\mu k + 2\mu - \lambda + 1)} = z\, E_{2\mu,2\mu-\lambda+1}(-z^2),$$

$$\cos_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(2\mu k + \mu - \lambda + 1)} = E_{2\mu,\mu-\lambda+1}(-z^2),$$

• Here we mention also the so-called *Lorenzo-Hartley functions* (2000), the F-function and its generalization - the R-function, shown to be solutions of some linear fractional differential equations. We can represent them in terms of M-L function, namely, for $z > 0$, $c = 0$, $q \geq 0$, $\nu \leq q$:

$$F_q(a, z) = \sum_{k=0}^{\infty} \frac{a^k z^{(k+1)q-1}}{\Gamma((k+1)q)} = z^{q-1}\, E_{q,q}(az),$$

$$R_{q,\nu}(a, 0, z) = \sum_{k=0}^{\infty} \frac{a^k z^{(k+1)q-1-\nu}}{\Gamma((k+1)q-\nu)} = z^{q-1}\, E_{q,q-\nu}(az).$$

Examples of multi-index M-L functions $E_{(\alpha_i),(\beta_i)}(z)$, $m > 1$

For $m = 2$:

• We start with the not enough popular *M-L type function of Dzrbashjan* (1960, in Russian only), with 2×2 indices, which he denoted alternatively by (set $1/\rho_i := \alpha_i, \mu_i := \beta_i$, $i = 1, 2$):

$$\Phi_{\rho_1,\rho_2}(z; \mu_1, \mu_2) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1})\Gamma(\mu_2 + \frac{k}{\rho_2})} \tag{20}$$
$$:= E_{(\frac{1}{\rho_1},\frac{1}{\rho_2}),(\mu_1,\mu_2)}(z) = E_{(\alpha_1,\alpha_2),(\beta_1,\beta_2)}(z).$$

Dzrbashjan found its order and type as entire function, claimed on few simple particular cases, and considered some integral relations between (20) and Mellin transforms on a set of axes. Then, he developed a theory of integral transforms in the class L_2, involving kernel close to functions (20) and further, proposed approximations of entire functions in L_2 for an arbitrary finite system of axes in complex plane starting from the origin.

The 2×2-indices M-L type functions (20) are studied in details also by Luchko in recent works (as 2020). He allows the parameters ρ_1, ρ_2 to be also negative or zero, and called them *"4-parameters Wright functions of second kind"*.

Simple cases of (20) as mentioned by Dzrbashjan himself, were:

- the *M-L function* (itself):

$E_{\frac{1}{\rho},\mu}(z) = E_{(\frac{1}{\rho},0),(\mu,1)}(z) = \Phi_{\rho,\infty}(z;\mu,1)$; also:

$\dfrac{1}{1-z} = E_{(0,0),(1,1)}(z) = \Phi_{\infty,\infty}(z;1,1)$; the *Bessel function*:

$J_\nu(z) = \left(\dfrac{z}{2}\right)^\nu E_{(1,1),(\nu+1,1)}\left(-\dfrac{z^2}{4}\right) = \left(\dfrac{z}{2}\right)^\nu \Phi_{1,1}\left(-\dfrac{z^2}{4};1,\nu+1\right)$;...

To these examples, we have added (Kiryakova, 2010-2021) also:

- The *Struve and Lommel functions*:

$$s_{\mu,\nu}(z) = \frac{1}{4} z^{\mu+1} E_{(1,1),((3-\nu+\mu)/2,(3+\nu+\mu)/2)}\left(-\frac{z^2}{4}\right),$$

$$H_\nu(z) = \frac{1}{\pi 2^{\nu-1}(1/2)_\nu}\, s_{\nu,\nu}(z).$$

- The (classical) *Wright function* studied by Fox (1928), Wright (1933), Humbert and Agarwal (1953), extended also for $\alpha > -1$, is also a case of the multi-M-L function with $m = 2$:

$$\varphi(\alpha,\beta;z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} = {}_0\Psi_1\left[\begin{array}{c} - \\ (\beta,\alpha) \end{array} \middle| z\right] = E^{(2)}_{(\alpha,1),(\beta,1)}(z). \tag{21}$$

This Wright f. plays important role in the solution of linear PFDEs as the *fractional diffusion-wave equation* studied by Nigmatullin (1984-1986, to describe the diffusion process in media with fractal geometry, $0 < \alpha < 1$) and by Mainardi et al. (1994 -), for propagation of mechanical diffusive waves in viscoelastic media, $1 < \alpha < 2$). In the form $M(z;\beta) = \varphi(-\beta, 1-\beta; -z)$, $\beta := \alpha/2$, it is called also as the *Mainardi function*. In our denotations, this is: $M(z;\beta) = E^{(2)}_{(-\beta,1),(1-\beta)}(-z)$, $m = 2$ and has its examples like: $M(z;1/2) = 1/\sqrt{\pi}\,\exp(-z^2/4)$ and the *Airy function*: $M(z;1/3) = 3^{2/3}\,Ai(z/3^{1/3})$.

In other form and denotation, the same Wright function is known as *Wright-Bessel*, or misnamed as *Bessel-Maitland function*:

$$J^{\mu}_{\nu}(z) = \varphi(\mu, \nu{+}1; -z) = {}_0\Psi_1 \left[\begin{array}{c} - \\ (\nu+1,\mu) \end{array} \middle| -z \right]$$

$$= \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\nu + k\mu + 1)\,k!} = E^{(2)}_{(1/\mu,1),(\nu+1,1)}(-z)\,, \qquad (22)$$

again as an example of the Dzrbashjan function. It is an obvious *"fractional index" analogue of the classical Bessel function $J_\nu(z)$*.

Nowadays, several further *"fractional-indices" generalizations of* $J_\nu(z)$ are exploited, and we can present them as multi-M-L functions! Such one is the so-called *generalized Wright-Bessel (-Lommel) functions*, due to Pathak (1966-1967):

$$\begin{aligned} J^{\mu}_{\nu,\lambda}(z) &= (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\nu+k\mu+\lambda+1)\Gamma(\lambda+k+1)} \\ &= (z/2)^{\nu+2\lambda} E^{(2)}_{(1/\mu,1),(\nu+\lambda+1,\lambda+1)} \left(-(z/2)^2\right), \ \mu > 0, \end{aligned} \tag{23}$$

including, for $\mu = 1$, the *Lommel* (and thus, also the *Struve*) *functions* $J^1_{\nu,\lambda}(z) = \text{const}\, S_{2\lambda+\nu-1,\nu}(z)$. Next one, is the *generalized Lommel-Wright function with 4 indices*, introduced by de Oteiza, Kalla and Conde, $r > 0, n \in \mathbb{N}, \nu, \lambda \in \mathbb{C}$:

$$\begin{aligned} J^{r,n}_{\nu,\lambda}(z) &= (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{k}}{\Gamma(\nu+kr+\lambda+1)\Gamma(\lambda+k+1)^n} \\ &= (z/2)^{\nu+2\lambda} E^{(n+1)}_{(1/r,1,\dots,1),(\nu+\lambda+1,\lambda+1,\dots,\lambda+1)} \left(-(z/2)^2\right), \end{aligned} \tag{24}$$

an interesting example of a multi-M-L function with $m = n + 1$.

Other special cases:

- For arbitrary $\underline{m \geq 2}$: let $\forall \rho_i = \infty$ $(1/\rho_i = 0)$ and $\forall \mu_i = 1$, $i = 1, ..., m$. Then, from definition of the multi-index M-L f.,

$$E_{(0,0,...,0),(1,1,...,1)}(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \, .$$

- Consider the case $m \geq 2$, with $\forall \rho_i = 1$, $i = 1, \ldots, m$. Then:

$$E^{(m)}_{(1,1,...,1),(\mu_i+1)}(z) = {}_1\Psi_m \left[\begin{array}{c} (1,1) \\ (\mu_i, 1)_1^m \end{array} \middle| z \right] = \text{const}\, {}_1F_m\,(1; \mu_1, \mu_2, ..., \mu_m; z)$$

reduces to ${}_1F_m$ - and to a *Meijer's* $G^{1,1}_{1,m+1}$-function. Denote $\mu_i = \gamma_i + 1, i = 1, \ldots, m$, and let additionally one of the μ_i to be 1, e.g.: $\mu_m = 1$, i.e. $\gamma_m = 0$. Then the multi-M-L function becomes a *hyper-Bessel function*, which is *integer* multi-index analogue of J_ν:

$$J^{(m-1)}_{\nu_i,...,\nu_{m-1}}(z) = \left(\frac{z}{m}\right)^{\sum_{i=1}^{m-1} \nu_i} E^{(m)}_{(1,1,...,1),(\nu_1+1,\nu_2+1,...,\nu_{m-1}+1,1)} \left(-(\frac{z}{m})^m\right) . \tag{25}$$

In view of the above relation, the multi-index M-L functions with arbitrary $(\alpha_1, \ldots, \alpha_m) \neq (1, \ldots, 1)$ can be seen as *fractional-indices analogues of the hyper-Bessel functions*.

The hyper-Bessel functions (25) themselves are *multi-index* (but integer) *analogues* of the Bessel function. Functions (25) are closely related to the *hyper-Bessel differential operators* of the form (mentioned in Lecture 1):

$$Bf(t) = t^{\alpha_0}\frac{d}{dt}t^{\alpha_1}\frac{d}{dt}\cdots t^{\alpha_{m-1}}\frac{d}{dt}t^{\alpha_m}f(t) = t^{-\beta}P_m\left(t\frac{d}{dt}\right)f(t)$$
$$= t^{-\beta}\prod_{k=1}^{m}\left(t\frac{d}{dt}+\beta\gamma_k\right)f(t), \quad t>0, \tag{26}$$

and form a fundamental system of solutions of the differential equations of the form $By(z)=\lambda y(z)$ (Kiryakova, 1994, Th.3.4.3). For example, if $\beta=m$, $\gamma_1<\gamma_2<...<\gamma_m=0<\gamma_1+1$ in (26), the solution of the Cauchy problem $By(z)=-y(z),\ y(0)=1,\ y^{(j)}(0)=0, j=1,...,m-1$, is given by the *normalized hyper-Bessel function*: $y(z)=j^{(m-1)}_{\gamma_1,..,\gamma_{m-1}}(-z)$.

Related also are the *Bessel-Clifford functions of m-th order*:

$$C_{\nu_1,\ldots,\nu_m}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{\Gamma(\nu_1+k+1)\ldots\Gamma(\nu_m+k+1)\,k!}$$
$$= E^{(m+1)}_{(1,\ldots,1),(\nu_1+1,\ldots,\nu_m+1,1)}(-z).$$

Then, let us mention the special functions appearing in some recent papers by *Ricci* (say, 2020). He considers the so-called *Laguerre derivative* $D_L = \frac{d}{dz} z \frac{d}{dz}$ and its iterates $D_{mL} = \frac{d}{dz} z \frac{d}{dz} z ... \frac{d}{dz} z$. But these are the same as the particular hyper-Bessel differential operators considered in operational calculus by Ditkin and Prudnikov (1963). Then, the *L-exponentials* $e_1(z), e_2(z), ..., e_m(z), ...$ which are eigenfunctions of D_{mL}, that is, $D_{mL}\, e_m(\lambda z) = \lambda\, e_m(\lambda z)$, have been shown by Ricci to have the form $e_m(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^{m+1}}$, but hese can be seen to be

$$e_m(z) = {}_0F_m(-; 1, 1, ..., 1; z) = {}_1\Psi_{m+1}\left[\begin{array}{c} (1,1) \\ (1,1), (1,1), ..., (1,1) \end{array} \middle| z\right].$$

Thus, these are examples of both hyper-Bessel functions and multi-index M-L functions $E^{(m+1)}_{(1,...,1),(1,...,1)}(z)$. Ricci applied these SF and the related Laguerre-type generalized hypergeometric functions as solutions in population dynamics.

• One may consider *multi-index analogues of the Rabotnov (α-exponential function)*, with all $\mu_i = 1/\rho_i = \alpha > 0, i = 1, ..., m$:

$$y_\alpha^{(m)}(z) = z^{\alpha-1} E^{(m)}_{(\alpha,..,\alpha),(\alpha,..,\alpha)}(z^\alpha) = z^{\alpha-1} \sum_{k=0}^{\infty} \frac{z^{\alpha k}}{[\Gamma(\alpha+\alpha k)]^m}, \quad (27)$$

and for $\alpha = 1$ we have : $\sum_{k=0}^{\infty} \frac{z^k}{[k!]^m}$, the original of Le Roy function!

• In general, for rational values of $\forall \alpha_i, i = 1, ..., m$, the multi-index M-L functions $Ei(z)$ are reducible to *Meijer's G-functions* (that is, to classical special functions!).

• As a special case of the $3m$-parametric M-L functions $E^{(\gamma_i),m}_{(\alpha_i),(\beta_i)}(z)$, aside from the Prabhakar function with $m = 1$, we can mention the *M-L type f.* by Kilbas-Srivastava-Trujillo, when $\gamma_1 := \gamma$, but all rest $\gamma_i = 1$, $i = 2, ..., m$:

$$E^\gamma((\alpha_i, \beta_i)_{1,m}; z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\prod_{j=1}^{m} \Gamma(\alpha_i k + \beta_i)} \frac{z^k}{k!}.$$

Other special cases of the Wright generalized hypergeometric functions ${}_p\Psi_q$

- *Virchenko and Ricci generalized hypergeometric functions*:

Virchenko (1999, and on) studied some generalized hypergeometric functions denoted by ${}_2R_1^{\tau}(z)$ and ${}_1\Phi_1^{\tau}(z)$, and their integral representations, relations and applications to the generalized Legendre functions $P_k^{m,m}(z), Q_k^{m,n}(z)$, gamma functions, Laguerre's functions, etc.:

$${}_2R_1^{\omega,\mu}(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+\frac{\omega}{\mu}k)}{\Gamma(c+\frac{\omega}{\mu}k)} \cdot \frac{z^k}{k!}.$$

For $\dfrac{\omega}{\mu} := \tau > 0$, and a, b, c - complex, $a+k \neq 0, -1, -2, \ldots;$ $b+\tau k \neq 0, -1, -2, \ldots, k = 0, 1, 2, \ldots;\ |z| < 1$, it is rewritten as

$${}_2R_1^{\tau}(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)} \cdot \frac{z^k}{k!}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\, {}_2\Psi_1 \left[\begin{array}{c} (a,1),(b,\tau) \\ (c,\tau) \end{array} \middle| z \right].$$

Virchenko proposed also some examples of elementary functions for these special functions, as: $(\ln(1+z))_\tau$, $(\arcsin z)_\tau$,..., some generalized incomplete B-function; the Gauss function ${}_2F_1$, etc.

The other function studied by Virchenko, is:

$$ {}_1\Phi_1^\tau(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+\tau k)}{\Gamma(c+\tau k)} \cdot \frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)}\, {}_1\Psi_1 \left[\begin{array}{c} (a,\tau) \\ (c,\tau) \end{array} \middle| z \right], $$

and this includes generalizations of the gamma function, incomplete gamma function, probability integrals and Laguerre's functions that are confluent type g.h.f.

- In the same recent work, *Ricci* mentioned about Laguerre-type derivatives and related special functions. He considered also some Laguerre-type (L-) Bessel functions, L-type Gauss hypergeometric functions, and the *Laguerre-type generalized hypergeometric functions* ${}_{Lp}F_q$ which can be shown to be representable by ${}_pF_{q+1}$:

$$ {}_{Lp}F_q(a_1,..,a_p;b_1,...,b_q;z) = \sum_{k=0}^{\infty} \frac{a_1^{(k)}...a_p^{(k)}}{b_1^{(k)}...b_q^{(k)}} \cdot \frac{z^k}{(k!)^2} $$

$$ = \sum_{k=0}^{\infty} \frac{a_1^{(k)}...a_p^{(k)}}{b_1^{(k)}...b_q^{(k)}\,(1)^{(k)}} \cdot \frac{z^k}{k!} = {}_pF_{q+1}(a_1,...,a_p;b_1,...,b_q,1;z). $$

- *Mainardi-Masina and Paris generalized exponential integrals*:

Mainardi and Masina (2020) introduced a generalized exponential integral $\mathrm{Ein}_\alpha(z)$ by replacing the exponential function in the complementary exponential integral $\mathrm{Ein}(z)$ by the Mittag-Leffler function $E_\alpha(z)$, with physical applications for $0 < \alpha < 1$ in the studies of the creep features of a linear viscoelastic models. Also recently, *Paris* (2020) made the next step to involve the 2-parameters M-L function, namely to consider the generalized exponential integral

$$\mathrm{Ein}_{\alpha,\beta}(z) = z \sum_{k=0}^{\infty} \frac{(-1)^k z^{\alpha k}}{(ak+1)\Gamma(\alpha k + \alpha + \beta)} \quad (\text{for } \beta = 1: \ \mathrm{Ein}_\alpha(z)). \tag{29}$$

This function can be seen as a case of the Wright g.h.f. with $p = q = 2$, namely

$$\begin{aligned} \mathrm{Ein}_{\alpha,\beta}(z) &= z \sum_{k=0}^{\infty} \frac{\Gamma(\alpha k + 1)\Gamma(k+1)}{\Gamma(\alpha k + 2)\Gamma(\alpha k + \alpha + \beta)} \frac{(-z^\alpha)^k}{k!} \\ &= z \, {}_2\Psi_2 \left[\begin{matrix} (1,\alpha),(1,1) \\ (2,\alpha),(\alpha+\beta,\alpha) \end{matrix} \middle| -z^\alpha \right]. \end{aligned}$$

Paris studied in details its asymptotic expansion for $|z| \to \infty$.

Some "new" but NOT new special functions: k- and S-variants ...

- The so-called *k-analogues*:

Claims on inventing and studying "new" classes of special functions in many recent papers have been based on the extended notion of the *k-Gamma function*, $k > 0$. However, in all such works its representation in terms of the classical Gamma-function $\Gamma(.)$ is explicitly written there, although ignored:

$$\Gamma_k(s) = \int_0^\infty \exp(-\frac{t^k}{k})\, t^{s-1} dt = k^{\frac{s}{k}-1}\, \Gamma(\frac{s}{k}), \quad s \in \mathbb{C}, \mathrm{Re}\,(s) > 0. \tag{30}$$

Also, the *k-Pochhamer symbol* is used in the denotations:

$$(\lambda)_{\nu,\kappa} := \Gamma_k(\lambda + \nu\kappa)/\Gamma_k(\lambda), \quad \lambda \in \mathbb{C} \setminus \{0\}, \ \nu \in \mathbb{C}. \tag{31}$$

In Kiryakova (2019), using the above definitions, we have shown that *most of these "new" functions are in fact some known special functions, namely Wright g.h.f. and its cases*. Also there, and in Kiryakova (2020-2021), in References lists are given dozens of discussed particular authors/ sources.

More deep results on SF of FC:

• An unified approach for evaluation of images of the SF and SF of FC under operators of FC and GFC, has been proposed in Kiryakova (2017-2021) in the most general case.

• Also, theory of classification of SF (in general, ${}_pF_q$) and of SF of FC (as in general, are ${}_p\Psi_q$) in 3 basic classes, depending on if $p < q$, $p = q$ or $p = q + 1$, is proposed. This helps to facilitate the comprehension of the SF for not close experts but users, as: "cos-Bessel", "exp-confluent" and "geom.series-Gauss h.f."-types.

Numerical aspects for SF of FC:

Several comments and short information (what our FC colleagues as Marichev, Mainardi, Podlubny, Garrappa, Luchko, etc. have done, at least for few cases of the SF of FC) can be found in **Section 9 of the survey**:

V. Kiryakova, A Guide to Special Functions in Fractional Calculus, *Mathematics*, Vol. 9, No 1 (2021), Art. 106, 35 pp., https://doi.org/10.3390/math9010106; Open Access (MDPI).

References: V. Kiryakova (1994 – 2022), see next slides

Some References (authored by V. Kiryakova):

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