Parametric resonances for the wave equation with time periodic potential

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1. Introduction

Consider the ODE

$$x''(t) + \omega^2(1 + \epsilon q(t))x(t) = 0,\ 0 < \epsilon \ll 1 \tag{1}$$

with smooth period function $q(t + T) = q(t) \geq 0$. The eigenvalues of the propagator

$$A : (x(0), x'(0)) \longrightarrow (x(T), x'(T))$$

play a crucial role for the stability/instability of the solutions as $t \to +\infty$. For $\epsilon = 0$ we have globally bounded solutions, while for $\epsilon > 0$ there exist intervals $I_j = (\alpha_j(\epsilon), \beta_j(\epsilon)) \subset \mathbb{R}^+$ such that for $\omega \in I_j$ the operator A has eigenvalue $\mu_j, |\mu_j| > 1$ with eigenfunction $X_j(0) = (x_{j,0}, x_{j,1})$. The solution $x(t)$ of (1) with initial data $X_j(0)$ is exponentially increasing as $t \to +\infty$. This phenomenon is called *parametric resonance*.

If we add a nonlinear term the situation is completely different. For example, consider the equation

$$x''(t) + x^{2m+1}(t) + \sum_{j=0}^{2m} p_j(t)x^j(t) = 0 \tag{2}$$

with periodic functions $p_j(t+1) = p_j(t)$, $j = 0, 1, ..., 2m$. R. Dieckerhoff and E. Zehnder (1987) proved that all solutions of (2) exist for $t \in \mathbb{R}$ and for every solution we have

$$\sup_{t\in\mathbb{R}}(|x(t)| + |x'(t)|) < \infty.$$

The strategy is to write the equation as a Hamiltonian system with time dependent Hamiltonian $H(x, y, t) = \frac{y^2}{2} + V(x, t)$ and after several transformations to apply the Moser twist theorem if $|x(t_0)| + |x'(t_0)|$ is sufficiently large.

Passing to wave equation, consider the linear operator

$$Pu = \partial_t^2 u - \Delta_x u + q(t,x)u,$$

where $0 \leq q(t,x) \in C^\infty(\mathbb{R}_t \times \mathbb{R}^3)$ is periodic in time t with period $T > 0$ and has a compact support with respect to x included in $\{x \in \mathbb{R}^3 : |x| \leq \rho\}$. Let $u(t,x;s)$ be the solution of the Cauchy problem

$$Pu = 0,\ u(s,x) = f_1(x),\ u_t(s,x) = f_2(x) \tag{3}$$

with $f = (f_1, f_2) \in H = H_D(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, where $H_D(\mathbb{R}^3)$ is the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|f\|_{H_D} = \Big(\int |\nabla_x f|^2 dx\Big)^{1/2}.$$

Set $\|f\| = (\|f_1\|^2_{H_D} + \|f_2\|^2_{L^2})^{1/2}$, $\|f\|_1 = (\|f_1\|^2_{H^1} + \|f_2\|^2_{L^2})^{1/2}$.

Therefore the operator

$$H \ni f \to U(t,s)f = (u(t,x;s), u_t(t,x;s)) \in H$$

is called the propagator (monodromy operator) of (3) and there exist $C > 0$ and $\alpha \geq 0$ so that

$$\|U(t,s)f\| \leq Ce^{\alpha|t-s|}\|f\|. \tag{4}$$

Moreover, we have $U(t+T, s+T) = U(t,s)$ and $U(0,nT) = (U(0,T))^n$. If $U(0,T)$ has an eigenvalues μ, $|\mu| > 1$, with eigenfunction f, then $U(0,nT)f = \mu^n f$ has exponentially increasing energy.

It was conjectured that there exist positive time-periodic potentials q for which $U(0,T)$ has an eigenvalue $\mu, |\mu| > 1$, which is an analog to the parametric resonance for the wave equation. This conjecture has been proved in 2008 by Colombini, Petkov and Rauch. If we consider a smaller energy space $\mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \subset H$ the above result is not directly applied. We extend this result for $\mathcal{H}$, so for some q there are solutions $u(x,t)$ with initial data in $\mathcal{H}$ whose energy norm $\|u(t,x)\|_{H^1(\mathbb{R}^3)}$ is exponentially increasing.

Motivated by the results for ODE, it is natural to conjecture that adding a nonlinear defocusing term the parametric resonance disappear and the solutions of the Cauchy problem for the nonlinear wave equation

$$\begin{cases} u_{tt} - \Delta_x u + q(t,x)u + |u|^r u = 0, x \in \mathbb{R}^3, \; 2 \leq r < 4, \\ u(0,x) = f_1, u_t(0,x) = f_2 \end{cases} \tag{5}$$

have bounded or polynomially bounded energy norm as $t \to +\infty$. By using the continuous inclusion $H_D(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, it is easy to show that $f_1 \in H^1(\mathbb{R}^3)$ implies for $0 < \alpha < 4$

$$\int |f_1|^{\alpha+2} dx \leq \|f_1\|_1^{\alpha+2}.$$

In the following we will study the Cauchy problem (5) with initial data $(f_1, f_2) \in \mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

2.Results.

By a modification of the argument of [CPR], we obtain

Theorem 1

There exist $q(x,t) \geq 0$ periodic in time and $f = (f_1, f_2) \in \mathcal{H}$ such that the solution of (3) with initial data f satisfies :

$$\exists\, C > 0,\ \exists\, \alpha > 0 \quad \text{such that} \quad \forall\, t \geq 0, \quad \|u(t,\cdot)\|_{H^1(\mathbb{R}^3)} \geq C\, e^{\alpha t}\,. \tag{6}$$

In fact we show that the propagator

$$V(T,0) : \mathcal{H} \ni (f_1(x), f_2(x)) \longrightarrow (u(T,x), u_t(T,x)) \in \mathcal{H}$$

has an eigenvalue y, $|y| > 1$ which implies (6).

Theorem 2

For any choice of q the Cauchy problem (5) is globally well-posed in $\mathcal{H}$, that is for every $(f_1, f_2) \in \mathcal{H}$ there exists an unique solution $u(t,x) \in C^0(\mathbb{R}; H^1(\mathbb{R}^3)) \cap C^1(\mathbb{R}; L^2(\mathbb{R}^3))$. Moreover, there exists a constant $C > 0$ such that for every $t \in \mathbb{R}$, the solution $u(t,x)$ satisfies the polynomial bounds

$$\|\nabla_x u(t,\cdot)\|_{L^2(\mathbb{R}^3)} + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq 2\Big(X(0)^{\frac{r}{r+2}} + C|t|\Big)^{\frac{r+2}{2r}},$$

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq \|f_1\|_{L^2(\mathbb{R}^3)} + 2|t|\Big(X(0)^{\frac{r}{r+2}} + C|t|\Big)^{\frac{r+2}{2r}},$$

where

$$X(t) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}\Big(|\partial_t u|^2 + |\nabla_x u|^2 + q|u|^2\Big) + \frac{1}{r+2}|u|^{r+2}\Big) dx$$

and $C > 0$ depends only on q and r.

Here the solution is taken in the sense of distributions or as a solution of an integral equation. Notice that the solution $v(t,x)$ of the Cauchy problem of the free wave equation $v_{tt} - \Delta_x v = 0$ with initial data $f \in \mathcal{H}$ satisfies an estimate $\|v(t,x)\|_{H^1(\mathbb{R}^3)} \leq C(1+|t|)$.

It is classical to expect that the result of Theorem 1 implies the instability of the zero solution of (5). More precisely, we have the following instability result.

Theorem 3

With q as in Theorem 1 the following holds true. There is $\eta > 0$ such that for every $\delta > 0$ there exists $(f_1, f_2) \in \mathcal{H}$, $\|(f_1, f_2)\|_{\mathcal{H}} < \delta$ and there exists $n = n(\delta) > 0$ such that the solution of (5) satisfies $\|(u(nT, \cdot), \partial_t u(nT, \cdot)\|_{\mathcal{H}} > \eta$.

We may study also the growth of the $H^s(\mathbb{R}^3)$, $s \geq 2$, norms of the solutions of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta_x u + q(t,x)u + u^3 = 0, \\ (u(0,x), u_t(0,x)) = (f_1, f_2) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3). \end{cases} \quad (7)$$

Theorem 4

For any choice of q the Cauchy problem (7) has an unique solution $u(t,x) \in C^0(\mathbb{R}; H^s(\mathbb{R}^3)) \cap C^1(\mathbb{R}; H^{s-1}(\mathbb{R}^3))$. Moreover, there exist constant $C_s > 0$ and $m_s > 1$ depending on $\|(f_1, f_2)\|_{\mathcal{H}(\mathbb{R}^3)}$, q and s such that for every $t \in \mathbb{R}$, the solution $u(t,x)$ satisfies the polynomial bound

$$\|u(t,x)\|_{H^s(\mathbb{R}^3)} + \|u_t(t,x)\|_{H^{s-1}(\mathbb{R}^3)} \leq C_s \Big(1 + |t|\Big)^{m_s}, \; t \in \mathbb{R}. \quad (8)$$

3. Idea for the polynomial bound of the global energy

To obtain a priori estimate of the $H^1(\mathbb{R}^3)$ norm of the solution of (5), we need the following simple lemma which is consequence of a more general lemma of Osgood.

Lemma 1

Let $0 < \gamma < 1$ *and let* $X(t) : [0, \infty) \to [0, \infty)$ *be a derivable function such that*

$$|X'(t)| \leq CX^{1-\gamma}(t),\ 0 \leq t \leq \tau. \tag{9}$$

Then

$$X(t) \leq (X^{\gamma}(0) + C\gamma t)^{\frac{1}{\gamma}},\ 0 \leq t \leq \tau.$$

We wish to apply this lemma for

$$X(t) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{1}{2}q|u|^2 + \frac{1}{r+2}|u|^{r+2}\Big)dx.$$

First we prove the following

Lemma 2

The solutions

$$u(t,x) \in C([0,A], H_x^2(\mathbb{R}^3)) \cap C^1([0,A], H_x^1(\mathbb{R}^3)) \cap L_t^{\frac{2r+2}{r-2}}([0,A], L_x^{2r+2}(\mathbb{R}^3))$$

of (5) satisfy the relation

$$\frac{d}{dt}X(t) = \frac{1}{2}\,\mathrm{Re}\int_{\mathbb{R}^3} q_t|u|^2dx,\ 0 \leq t \leq A. \tag{10}$$

Remark 1

We show that (10) *holds in the sense of distributions* $\mathcal{D}'(]0,A[)$. *Since the right hand side of* (10) *is continuous in* $]0,A[$ *the derivative of the left hand side can be taken in the classical sense.*

Now $q(t,x) = 0$ for $|x| > \rho$ implies

$$\Big|\int q_t |u|^2 dx\Big| \leq C\|u\|^2_{L^2(|x|\leq\rho)} \leq C_1\|u(t,.)\|^{\frac{2}{r+2}}_{L^{r+2}(|x|\leq\rho)}.$$

Therefore

$$|X'(t)| \leq C_2 X^{\frac{2}{r+2}}(t) = C_2 X^{1-\frac{r}{r+2}}(t),$$

and applying Lemma 1 with $\gamma = \frac{r}{r+2} < 1$, we deduce

$$X(t) \leq (X^{\frac{r}{r+2}}(0) + \frac{C_2 r}{r+2} t)^{\frac{r+2}{2}},\ 0 \leq t \leq A, \tag{11}$$

hence

$$\|u_t(t,x)\|_{L^2} + \|\nabla_x u(t,x)\|_{L^2} \leq 2(X^{\frac{r}{r+2}}(0) + C_3 t)^{\frac{r+2}{4}}. \tag{12}$$

Next from

$$u(t,x) = u(0,x) + \int_0^t u_t(\tau, x) d\tau$$

one obtains

$$\|u(t,x)\|_{L^2} \leq \|u(0,x)\|_{L^2} + 2t(X^{\frac{r}{r+2}}(0) + C_3 t)^{\frac{r+2}{4}},\ 0 \leq t \leq A. \tag{13}$$

4. Global existence of solution

For the proof of global existence there are 2 difficulties:

1) The argument in the previous section works if we know that the solution exists for $0 \leq t \leq A$.

2) Lemma 2 is proved only for "smooth" solutions. The continuous dependence on initial data works only for "small" in time intervals and the zero solution could be unstable as it was shown in Theorem 3.

By using the argument of Petkov, one may show that the solution of (3) satisfies the same *local in time* Strichartz estimates as in the case $q = 0$. For these local estimates we don't need a global control of the local energy. More precisely, for every finite $a > 0$ and $f = (f_1, f_2) \in \mathcal{H}, F \in L^1([s, s + a]; L^2(\mathbb{R}^3))$ the solution of (3) satisfies

$$\|(u, \partial_t u)\|_{C([s,s+a];\mathcal{H})} + \|u\|_{L^p_t([s,s+a],L^q_x(\mathbb{R}^3))} \leq C(a)\Big(\|(f_1, f_2)\|_{\mathcal{H}} + \|F\|_{L^1([s,s+a];L^2(\mathbb{R}^3))}\Big), \tag{14}$$

provided $\frac{1}{p} + \frac{3}{q} = \frac{1}{2}$, $p > 2$ (the constant $C(a)$ in (14) depends on a, p and $q(t, x)$). Moreover, if $(f_1, f_2) \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$, $k \geq 2$ and $F \in L^1([s, s + a]; H^k(\mathbb{R}^3))$, we have

$$\|(u, \partial_t u)\|_{C([s,s+a];H^k\times H^{k-1})} + \|\nabla_x u\|_{L^p_t([s,s+a],L^q_x(\mathbb{R}^3))} \leq C_k(a)\Big(\|(f_1, f_2)\|_{H^k\times H^{k-1}} + \|F\|_{L^1([s,s+a];H^k(\mathbb{R}^3))}\Big). \tag{15}$$

Consider the Cauchy problem

$$\partial_t^2 u - \Delta_x u + q(t,x)u + |u|^r u = 0,\ u(s,x) = f_1(x),\ \partial_t u(s,x) = f_2(x), \quad 2 \le r < 4. \quad (16)$$

By using a fixed point argument and Strichartz estimates, we prove

Proposition 1

There exist $C > 0$, $c > 0$ and $\gamma > 0$ such that for every $(f_1, f_2) \in \mathcal{H}$ there is a unique solution $(u, \partial_t u) \in C([s, s+\tau], H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))$ of (16) on $[s, s+\tau]$ with $\tau = c(1 + \|(f_1, f_2)\|_{\mathcal{H}})^{-\gamma}$. Moreover, the solution satisfies

$$\|(u, \partial_t u)\|_{C([s,s+\tau];\mathcal{H})} + \|u\|_{L_t^{\frac{2r+2}{r-2}}([s,s+\tau], L_x^{2r+2}(\mathbb{R}^3))} \le C\|(f_1, f_2)\|_{\mathcal{H}}\,. \quad (17)$$

If in addition $(f_1, f_2) \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$, $k \ge 2$, then $(u, \partial_t u) \in C([s, s+\tau_k]; H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3))$, where $\tau_k = c_k(1 + \|(f_1, f_2)\|_{\mathcal{H}})^{-\gamma}$.

Remark 2

The constant τ_k depends on k, q and the norm $\|f\|_{\mathcal{H}}$ of the initial data.

Passing to global existence, fix a number $a > 0$ and introduce the number

$$B_a := \|f\|_{\mathcal{H}} + a(B_1 + B_2 a)^{\frac{r+2}{2r}},$$

where $B_1 > 0$ and $B_2 > 0$ depend only on $\|f\|_{\mathcal{H}}$ and r. This number should be a bound of the energy of the solution $u(t, x)$ in $[0, a]$ with initial data $f \in \mathcal{H}$ if the above argument based on Lemma 1 and Lemma 2 works.
Define $\tau(a) := c(1 + B_a)^{-\gamma} < 1$ with the constants $c > 0, \gamma > 0$ of Proposition 1 and observe that the local existence theorem can be applied in the interval $[s, s + \tau(a)] \subset [0, a]$ if the norm of the initial data for $t = s$ is bounded by B_a.

Below we treat the case $2 < r < 4$ (the case $r = 2$ can be covered without using Strichartz estimates). Let $C_a > 0$ be a constant such that the local Strichartz estimates in $[0, a]$ are true with $C = C_a$. Let $p = \frac{2r+2}{r-2}$, $q = 2r + 2$, so $\frac{1}{p} + \frac{3}{q} = \frac{1}{2}$. Choose $0 < \epsilon(a) \leq \tau(a)$ small so that $D_r C_a^{r+1}(B_a + 1)^r \epsilon(a)^{(2-\frac{r}{2})} \leq \frac{1}{2}$, where D_r is a constant depending only on r.

Proposition 2

Choose a sequence $g_n = ((g_n)_1, (g_n)_2) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ *converging in* $\mathcal{H}$ *to* $(f_1, f_2) \in \mathcal{H}$, $\|(f_1, f_2)\|_{\mathcal{H}} \leq B_a$ *as* $n \to \infty$ *and let* $w_n(t, x)$ *be the solution of the problem (16) in the same interval* $[s, s + \epsilon(a)]$ *with initial data* g_n *for* $t = s$. *Let* $u(t, x)$ *be the solution with data* (f_1, f_2) *on* $t = s$. *Then*

$$\int_{\mathbb{R}^3} \Big(\frac{1}{2}\Big(|\partial_t w_n|^2 + |\nabla_x w_n|^2 + q|w_n|^2\Big) + \frac{1}{r+2}|w_n|^{r+2}\Big) dx$$

$$\longrightarrow_{n\to\infty} \int_{\mathbb{R}^3} \big(\frac{1}{2}(|\partial_t u|^2 + |\nabla_x u|^2 + q|u|^2) + \frac{1}{r+2}|u|^{r+2}\big) dx$$

in the sense of distributions $\mathcal{D}'(s, s + \epsilon(a))$.

We apply Prop.1 and Prop. 2 in the interval $[0, \epsilon(a)]$ and using Lemma 1 and 2, we deduce that the norm $\|u(t,x)\|_{H^1(\mathbb{R}^3)}$ is bounded by B_a for $t \in [0, \epsilon(a)]$.
Next we pass to the second step in the interval $[\epsilon(a), 2\epsilon(a)] \subset [0, a]$. By the local existence we have solution in $[\epsilon(a), 2\epsilon(a)]$ and $u(t,x)$ is defined in $[0, 2\epsilon(a)]$. On the other hand, we may approximate the initial data $(u(\epsilon(a), x), u_t(\epsilon(a), x))$ by functions $g_n^{(2)} \in H^2 \times H^1$ and by the above argument the solution $u(t,x)$ in $[\epsilon(a), 2\epsilon(a)]$ is approximated by solutions $w_n^{(2)}(t,x)$ for which (10) holds for $\epsilon(a) \leq t \leq 2\epsilon(a)$. Thus (10) is satisfied for $u(t,x)$ for $\epsilon(a) \leq t < 2\epsilon(a)$ and combining this with the first step, one concludes that the same is true for $0 \leq t \leq 2\epsilon(a)$. This makes possible to apply Lemma 1 for $0 \leq t \leq 2\epsilon(a)$ and to deduce (10), (11) with uniform constants leading to a bound by B_a. We can iterate this procedure, since $\tau(a), \epsilon(a), B_a$ depend only on $\|f\|_{\mathcal{H}}$, a and r. The solution $u(t,x)$ will be defined globally in a interval $[0, \alpha(a)]$ with $0 < a - \alpha(a) < \epsilon(a)$. Since $\alpha(a) > a - \epsilon(a) > a - 1$ and a is arbitrary, we have a global solution $u(t,x)$ defined for $t \geq 0$ with corresponding bound of the energy.

5. Growth of $H^2(\mathbb{R}^3)$ norms

For the analysis of H^2 norms it is not possible to apply the above argument since it is difficult to find an energy $X(t)$ such that its derivative is bounded by $CX^\gamma(t)$, $0 < \gamma < 1$. For this reason we apply another strategy based on the following

Lemma 3

Let $\{\alpha_n\}$ be a sequence of non-negative numbers such that with some constants $0 < \gamma < 1$, $C > 0$ and $y \geq 0$ we have

$$\alpha_n \leq \alpha_{n-1} + C((\alpha_{n-1})^{1-\gamma} + 1)(1+n)^y, \; n \geq 1.$$

Then there exists a constant $\tilde{C} > 0$ such that

$$\alpha_n \leq \tilde{C}(1+n)^{\frac{1+y}{\gamma}}, \; n \geq 1. \tag{18}$$

Let
$$(u(t,x), u_t(t,x)) \in C([s, s+\tau], H^2(\mathbb{R}^3)) \times C([s, s+\tau], H^1(\mathbb{R}^3)),$$
where $u(t,x)$ is the solution for $t \in [s, s+\tau_2]$ of the Cauchy problem for
$$u_{tt} - \Delta_x u + q(t,x)u + u^3 = 0.$$
Taking the derivative $\partial_{x_j} = \partial_j$, $j = 1,2,3$ and noting $u_j = \partial_j u$, $u_{jt} = \partial_j \partial_t u$, one gets in the sense of distributions
$$(u_{jt})_t - \Delta_x u_j + (\partial_j q)u + q u_j + 3u^2 u_j = 0. \tag{19}$$
It is easy to see that $(\partial_j q)u + q u_j + 3u^2 u_j \in C([s, s+\tau], L^2(\mathbb{R}^3))$. Therefore
$$(u_{jt})_t - \Delta_x u_j \in C([s, s+\tau], L^2(\mathbb{R}^3)).$$
Multiplying the equality (19) by u_{jt}, with integration by parts we have
$$\int \Big((u_{jt})_t - \Delta_x u_j\Big) u_{jt} dx = -\int (\partial_j q) u u_{jt} dx - \int q u_j u_{jt} dx - 3\int u^2 u_j u_{jt} dx. \tag{20}$$

After an approximation argument and an integration by parts, (20) can be written as

$$= \frac{1}{2}\partial_t \sum_{j=1}^{3}\Big[\int \Big((u_{jt})^2 + |\nabla_x(u_j)|^2 + 3u^2u_j^2 + qu_j^2\Big)(t,x)dx\Big] =$$

$$= -\sum_{j=1}^{3}\int (\partial_j q)uu_{jt}dx + 3\sum_{j=1}^{3}\int uu_t u_j^2 dx + \sum_{j=1}^{3}\frac{1}{2}\int q_t u_j^2 dx = I_1(t) + I_2(t) + I_3(t), \quad (21)$$

where the derivative ∂_t is taken in the sense of distributions. Introduce

$$X(t) := \frac{1}{2}\sum_{j=1}^{3}\Big[\int \Big((u_{jt})^2 + |\nabla_x(u_j)|^2 + 3u^2u_j^2 + qu_j^2\Big)(t,x)dx\Big].$$

Therefore $X(s+\tau) = X(s) + \sum_{j=1}^{3}\int_s^{s+\tau} I_j(t)dt$. The estimates of $I_1(t)$ and $I_3(t)$ are easy. For example,

$$\int (u_j)^2 dx \leq C_1(1+|t|)^2, \quad \int_s^{s+\tau} I_3(t)dt \leq C_2(1+s)^2.$$

To estimate the integral of $I_2(t)$, let $0 < \epsilon \ll 1$ be a small number. First by the generalised Hölder inequality one estimates

$$|I_2(t)| \leq 3\|u(t,x)\|_{L^q(\mathbb{R}^3)}\|u_t(t,x)\|_{L^{2+\epsilon}(\mathbb{R}^3)}\|u_j(t,x)\|^2_{L^4(\mathbb{R}^3)}$$

$$\leq 3\|u(t,x)\|_{L^q(\mathbb{R}^3)}\|u_t(t,x)\|_{L^{2+\epsilon}(\mathbb{R}^3)}\|u_j(t,x)\|^{1/2}_{L^2(\mathbb{R}^3)}\|u_j(t,x)\|^{3/2}_{L^6(\mathbb{R}^3)},$$

where

$$\frac{1}{q} = \frac{\epsilon}{4+2\epsilon}, \ \frac{1}{q} + \frac{1}{2+\epsilon} + \frac{1}{2} = 1.$$

For the integral with respect to t one applies the Hölder inequality and for small ϵ we have

$$\Big|\int_s^{s+\tau} I_2(t)dt\Big| \leq C_1\tau^{1/p'}(1+s)^{\frac{5}{2}+\epsilon}\|u(t,x)\|_{L^p([s,s+\tau];L^q_x(\mathbb{R}^3))}\Big(X(s)^{\frac{3}{4}+\frac{3\epsilon}{8}}+1\Big),$$

where

$$\frac{1}{p} + \frac{3\epsilon}{4+2\epsilon} = \frac{1}{2}, \ \frac{1}{p'} + \frac{1}{p} = 1.$$

We apply the Strichartz estimate and polynomial bounds for the H^1 norm and deduce

$$\|u(t,x)\|_{L^p([s,s+\tau];L_x^q(\mathbb{R}^3))} \leq C(\epsilon)(1+s)^6.$$

Finally for $0 < \tau \leq 1$ with $y = \frac{17}{2} + \epsilon$ we have

$$\Big|\int_s^{s+\tau} I_2(t)dt\Big| \leq C'(\epsilon)\tau^{\frac{1}{p'}}\Big(X(s)^{\frac{3}{4}+\frac{3\epsilon}{8}} + 1\Big)(1+s)^y. \tag{22}$$

Let $a > 1$ be a fixed number. By local existence there exists a solution in $[s, s+\tau(a)] \subset [0,a]$ with initial data g on $t = s$, where $\tau(a) = c\Big((1 + \|f\|_{H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} + a(B_1 + B_2 a)\Big)^{-\gamma} < 1$, Setting $X(n\tau(a)) = \alpha_n,\ n \leq N(a) \leq \frac{a}{\tau(a)}$, one deduces

$$\alpha_n \leq \alpha_{n-1} + C_2(\alpha_{n-1}^{7/8} + 1)(1+n)^9.$$

We are in position to apply Lemma 3 and to obtain

$$X(N(a)\tau(a)) \leq \tilde{C}(1+N(a))^{\frac{80}{7}}$$

$$\leq \tilde{C}\Big(1+\frac{a}{c}\Big)^{\frac{80}{7}}\Big(1 + \|f\|_{H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} + a(B_1 + B_2 a)\Big)^{\frac{80}{7}\gamma}.$$

Thank you for the attention !