Semi-continuity of Oseledets subspaces and Pesin sets with exponentially small tails

Luchezar Stoyanov

Differential Equations and Mathematical Models in Economy
Conference in Memory of Iordan Iordanov
Sofia, February 2023

Introduction (cont.)

The theory of **dynamical systems** originated from

the qualitative study of differential equations.

The first exact definition of stability of motion described by trajectories of differential equations was given by **Aleksander Lyapunov** in his PhD Thesis in 1892.

He introduced two methods, the first of which was based on linearization of the equations of motion and this is what defined the quantities later called **Lyapunov exponents.**

Lyapunov exponents found a natural place into the theory of dynamical systems and also into the related to it **ergodic theory**.

Introduction (cont.)

In today's talk we will consider a certain kind of smooth dynamical systems.

Let $f : M \longrightarrow M$ be a C^2 Axiom A diffeomorphism
or the time-one map of a C^2 Axiom A flow
on a compact Riemannian manifold M,
and let X be a **basic set** for f,
i.e. a compact locally maximal invariant hyperbolic set for f.

We will consider $f : X \longrightarrow X$.

The **hyperbolicity** of f means that there exist constants $C > 0$ and $0 < \lambda < 1$ and an f- invariant decomposition of the tangent space $T_x M$, $x \in X$, as follows:

1) in the case of diffeomorphism f:

$$T_x M = E^u(x) \oplus E^s(x)$$

2) in the case of a flow:

$$T_x M = E^0(x) \oplus E^u(x) \oplus E^s(x)$$

into a direct sum of non-zero linear subspaces, where $E^0(x)$ is the one-dimension subspace determined by the direction of the flow at x and

$$\|df^n(u)\| \leq C\lambda^n \|u\|, \ \forall n \geq 0, \ n \in \mathbb{Z}, \ \forall u \in E^s(x),$$

$$\|df^n(u)\| \leq C\lambda^n \|u\|, \ \forall n \leq 0 \ , \ n \in \mathbb{Z}, \forall u \in E^u(x).$$

Invariant measures

A Borel probability measure μ on X is called ***f*-invariant** if $\mu(A) = \mu(f^n(A))$ for every Borel subset A of X and every integer n.

In that case for every integrable function h (real- or complex-valued) on X we have

$$\int_X h \, d\mu = \int_X (h \circ f) \, d\mu.$$

The **Borel measures** are the ones defined on the Borel σ-algebra of X, i.e. the smallest σ-algebra containing all open (and closed) subsets of X.

Multiplicative Ergodic Theorem (MET)– special case

Let $f : X \longrightarrow X$ be as defined earlier,
and let μ be an f-invariant Borel probability measure on X.

Assume that μ is **ergodic**, that is for any invariant Borel subset A of M we have either $\mu(A) = 0$ or $\mu(A) = 1$.

In this particular case we have the following Consequence of the famous **Oseledets (1968) Theorem:**

Theorem (MET)

There exists an f-invariant Borel subset $\mathcal{L}$ of X with $\mu(\mathcal{L}) = 1$ such that for every $x \in \mathcal{L}$ there exists a df-invariant decomposition

$$T_x M = E_1(x) \oplus E_2(x) \oplus \ldots \oplus E_k(x)$$

and numbers

$$\lambda_1 < \lambda_2 < \ldots < \lambda_k$$

such that:

(a) For all $v \in E_i(x) \setminus \{0\}$ and all $i = 1, \ldots, k$ we have

$$\lim_{|n| \to \infty} \frac{1}{n} \log \|df_x^n(v)\| = \lambda_i$$

(b) For every $\epsilon > 0$ there exists a Borel function $A_\epsilon : \mathcal{L} \longrightarrow (1, \infty)$, such that for any $x \in \mathcal{L}$ and any $i = 1, \ldots, k$ we have

$$\frac{\|v\|}{A_\epsilon(x)\, e^{|n|\epsilon}} \leq \frac{\|df_x^n(v)\|}{e^{n\lambda_i}} \leq A_\epsilon(x)\, e^{|n|\epsilon}\|v\| \qquad (1)$$

for all $v \in E_i(x)$ and all integers $n \in \mathbb{Z}$, and

$$e^{-\epsilon} \leq \frac{A_\epsilon(f(x))}{A_\epsilon(x)} \leq e^{\epsilon} \quad , \quad x \in \mathcal{L}. \qquad (2)$$

The numbers $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ are called **Lyapunov characteristic exponents**.

For flows we have $\lambda_i = 0$ for some i.

A function A_ϵ satisfying (2) is called an **ϵ-slow varying function**.

A **Lyapunov ϵ-regularity function** is a function $A_\epsilon(x)$ which satisfies both (1) and (2) in the Theorem.

Example

Let $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an invertible linear transformation defined by an $n \times n$ matrix A with integer entries which preserves $\mathbb{Z}^n$.

Then A induces an isomorphism $f : \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \longrightarrow \mathbb{T}^n$ which is a diffeomorphism with $df(0) = A$.

The Lyapunov exponents of f are $\lambda_i = \log |\chi_i|$, where χ_i are the eigenvalues of A.

f is an Anosov diffeomorphism iff $|\chi_i| \neq 1$ for all i.

Regularity Comments

Topologically $\mathcal{L}$ could be very small.

Barreira and Saussol (2000), and other people as well, gave various examples of hyperbolic flows for which the complement of $\mathcal{L}$ has Hausdorff dimension $= \mathrm{dim}(M)$.

Similar examples for Anosov diffeomorphisms were given by Barreira and Schmeling (2000).

In general no regularity properties of the **Oseledets subspaces** $E_i(x)$ are known, not even a simple continuity for the "most regular" systems e.g. Anosov flows.

Lyapunov charts

Significance of Lyapunov regularity functions $A_\epsilon(x)$

- controls the growth of $d\phi_t$ over each bundle E_i
- using it one defines a **Lyapunov chart** on $B(x, r_\epsilon(x))$ on which we have control of the iterations of the non-linear map f similar to these in (1) for the linear map df:

Non-linear estimates

$$\frac{d(y,z)}{A_\epsilon(x)\, e^{|n|\epsilon}} \leq \frac{d(f^n(y), f^n(z))}{e^{n\,\lambda_i}} \leq A_\epsilon(x)\, e^{|n|\epsilon}\, d(y,z)$$

whenever $y = \exp_x(v), z = \exp_x(w) \in B(x, r_\epsilon(x))$ for some $v, w \in E_i(x)$ and $f^n(y), f^n(z) \in B(f^n(x), r_\epsilon(x))$, $n \in \mathbb{Z}$. Here $\exp_x : T_x M \longrightarrow M$ is an appropriately defined **exponential map** (only defined locally near 0).

Pesin sets

The regularity functions A_ϵ are in general **only measurable**, and it seems very little is known about them.

The so called **Pesin (regular) sets**

$$\mathcal{R}_\ell = \{x \in \mathcal{L} : A_\epsilon(x) \leq \ell\} \quad , \quad \ell \geq 1,$$

are of particular importance since on such sets we have 'uniform' non-linear estimates.

Clearly

$$\mu(\cup_{\ell=1}^{\infty} \mathcal{R}_\ell) = 1.$$

However so far there has been **no information about the measures** of the sets $\mathcal{R}_\ell$ and 'how quickly' they fill in $\mathcal{L}_0$.

Definition (Gouëzel - St. 2019)

A Pesin set P will be called a **Pesin set with exponentially small tails** for μ if for every $\delta > 0$ there exist constants $C, c > 0$ such that

$$\mu(\{x \in \mathcal{L} : \#\{m : 0 \leq m \leq n-1\ ,\ f^m(x) \notin P\} \geq \delta\, n\}) \leq C\, e^{-cn}$$

for every integer $n \geq 0$.

In other words, for every $n \geq 0$, removing a set of exponentially small measure from $\mathcal{L}$, for the remaining x, at least $(1-\delta)n$ points from the orbit $x, f(x), \ldots, f^{n-1}(x)$ belong to the Pesin set P.

In [Gouëzel - St. 2019] various sufficient conditions were given for existence of Pesin sets with exponentially small tails.

Motivation

In [St - 2017] we considered contact Anosov flows on compact Riemannian manifolds and proved exponential mixing for Gibbs measures determined by Hölder continuous potentials that admit Pesin sets with exponentially small tails.

In fact a stronger result was obtained in [St-2017] – this concerns strong spectral estimnates of the so called **Ruelle transfer operators** which have a variety of other applications, apart from the study of decay of correlations.

Open Problem:

Do Gibbs measures for 'nice systems" (e.g. Anosov diffeomorphisms and flows) always admit a Pesin set with exponentially small tails?

Upper semi-continuity

Let $f : X \longrightarrow X$ be as defined earlier,
and let μ be an f-invariant Borel probability measure on X

Definition ([St-2022])

We will say that the Oseledets subspaces $E_i(x)$ related to μ
depend upper semi-continuously on x
if we can choose the set $\mathcal{L}$ in MET so that
for every Lyapunov exponent λ_i, every $x \in \mathcal{L}$,
every sequence $\{x_n\}$ in $\mathcal{L}$ converging to x,
and every sequence $\{u_n\}_{n=1}^{\infty}$ with $u_n \in E_i(x_n)$, $\|u_n\| = 1 \; \forall n$,
if there exists $u = \lim_{n\to\infty} u_n$, then $u \in E_i(x)$.

Main result

Theorem ([St-2022])

Under the assumptions made earlier, let μ be a Gibbs measure on X so that the Oseledets subspaces $E_i(x)$ depend upper semi-continuously on x. Then there exists a Pesin set with exponentially small tails for μ.

An example – open billiards

Let K be a subset of $\mathbb{R}^n$ ($n \geq 2$) of the form

$$K = K_1 \cup K_2 \cup \ldots \cup K_{k_0},$$

where K_i are compact strictly convex disjoint domains in $\mathbb{R}^n$ with C^3 *boundaries* $\Gamma_i = \partial K_i$ and $k_0 \geq 3$.
We assume that K satisfies the following (no-eclipse) condition:

$$(\mathbf{H}) \quad \begin{cases} \text{for every pair } K_i, K_j \text{ of different connected components of } K \\ \text{the convex hull of } K_i \cup K_j \text{ has no common points with any other} \\ \text{connected component of } K. \end{cases}$$

With this condition, the *billiard flow* ϕ_t defined in a standard way in the exterior $\Omega = \mathbb{R}^n \setminus K$ of K is called an **open billiard flow**.
It has singularities, however its restriction to the **non-wandering set** Λ has only simple discontinuities at reflection points.

Open billiard (cont.)

The **non-wandering set** Λ for the flow ϕ_t is the set of those x in phase space such that the trajectory $\{\phi_t(x) : t \in \mathbb{R}\}$ is bounded.

It follows from results of Sinai that Λ is a hyperbolic set for ϕ_t, and it follows from the natural symbolic coding for the natural section of the flow (the so called billiard ball map) that the periodic points are dense in Λ and ϕ_t is transitive on Λ.

Thus, Λ is a basic set for ϕ_t.

Open billiard (cont.)

Theorem 2 ([St-2022])

Let $f : \Lambda \longrightarrow \Lambda$ be the time-one map for the open billiard flow on the non-wandering set Λ and let μ be an arbitrary Gibbs measure on Λ determined by a Hölder continuous potential on Λ. Then the Oseledets subspaces $E_i(x)$ depend upper semi-continuously on x.
Consequently, there exists a Pesin set with exponentially small tails for μ.

Open billiard (cont.)

Expected Consequences:

Results similar to these in [St - 2017] for contact Anosov flows should be obtained for open billiard flows.

References

L. Arnold, *Random Dynamical Systems*, Springer Monographs in Mathematics, Springer, Berlin 1998.

L. Barreira and Ya. Pesin, *Lyapunov exponents and smooth ergodic theory*, University Lecture Series 23, AMS 2001.

S. Gouëzel and L. Stoyanov, *Quantitative Pesin theory for Anosov diffeomorphisms and flows*, Ergodic Th. & Dyn. Sys. **39** (2019), 159-200.

V. I. Oseledets, *A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197-221.

Ya. Pesin, *Characteristic exponents and smooth ergodic theory*, Russian Mathematical Surveys **32** (1977), 55-114.

References (cont.)

L. Stoyanov – 2017, *Spectral properties of Ruelle transfer operators for regular Gibbs measures and decay of correlations for contact Anosov flows*, Memoirs Amer. Math. Soc. **1404** (2023); arXiv:1712.03103.

L. Stoyanov – 2022, *Semi-continuity of Oseledets subspaces and Pesin sets with exponentially small tails*; arXiv:2212.10237.